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Alternative Proof for the Localization of Sinai's Walk

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We give an alternative proof of the localization of Sinai's random walk in random environment under weaker hypothesis than the ones used by Sinai. Moreover, we give estimates that are stronger than the one of Sinai on the localization neighborhood and on the probability for the random walk to stay inside this neighborhood.

KEY WORDS: Random environment; random walk; Sinai's regime; Markov chain.

1. INTRODUCTION

Random Walks in Random Environment (R.W.R.E.) are basic processes in random media. The one dimensional case with nearest neighbor jumps, introduced by Solomon,[1] was first studied by Kesten *et al.*,[2] Sinai,[3] Golosov[4,5] and Kesten[6] all these works show the diversity of the possible behaviors of such walks depending on hypothesis assumed for the environment. At the end of the eighties Deheuvels and Révész[7] and Révész[8] give the first almost sure behavior of the R.W.R.E. in the recurrent case. Then we have to wait until the middle of the nineties to see new results. An important part of these new results concerns the problem of large deviations first studied by Greven and Hollander[9] and then by Zeitouni and Gantert,[10] Pisztora and Povel,[11] Zeitouni *et al.*[12] and Comets *et al.*[13] (see Zeitouni[14] for a review). In the same period using the stochastic calculus for the recurrent case Shi,[15] Hu and Shi,[16,17] Hu[18,19] and Hu

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and Shi[20] follow the works of Schumacher[21] and Brox[22] to give very precise results on the random walk and its local time (see Shi[23] for an introduction). Moreover, recent results on the problem of aging are given in Dembo *et al.*,[24] on the moderate deviations in Comets and Popov[25] for the recurrent case, and on the local time in Gantert and Shi[26] for the transient case. In parallel to all these results a continuous time model has been studied, see for example Schumacher[21] and Brox,[22] the works of Tanaka,[27] Mathieu,[28] Tanaka,[29] Tanaka and Kawazu,[30] Mathieu[31] and Taleb.[32]

Since the beginning of the eighties the delicate case of R.W.R.E. in dimension larger than 2 has been studied a lot, see for example Kalikow,[33] Anshelevich *et al.*,[34] Durett,[35] Bouchaud *et al.*,[36] and Bricmont and Kupianen.[37] For recent reviews (before 2002) on this topics see the papers of Sznitman[38] and Zeitouni.[14] See also Sznitman,[39] Varadhan,[40] Rassoul-Agha[41] and Zeitouni.[42]

In this paper we are interested in Sinai's walk i.e the one dimensional random walk in random environment with three conditions on the random environment: two necessaries hypothesis to get a recurrent process (see Solomon[1]) which is not a simple random walk and an hypothesis of regularity which allows us to have a good control on the fluctuations of the random environment.

The asymptotic behavior of such walk was discovered by Sinai,[3] he showed that this process is sub-diffusive and that at time n it is localized in the neighborhood of a well-defined point of the lattice. This *point of localization* is a random variable depending only on the random environment and n, its explicit limit distribution was given, independently, by Kesten[6] and Golosov.[5]

Here we give an alternative proof of Sinai's results under a weaker hypothesis. First we recall an elementary method proving that for a given instant n Sinai's walk is trapped in a basic valley denoted $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$ depending only on n and on a realization of the environment. Then we give a proof of the localization, this proof is based on an analysis of the return time to $\tilde{m}_0$. We get a stronger result than Sinai: we find that a size of the neighborhood of the localization depends on n like $(\log_2 n)^{9/2}(\log n)^{3/2}$ instead of $\delta(\log n)^2$ found by Sinai. Moreover, we compute the rates of the convergence of the probabilities (for the random walk and the random environment). Our method is based on the classification of the valleys obtained by ordered refinement of the basic valley $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$. The properties of the valleys obtained by this operation are proved with some details.

This paper is organized as follows. In Section 2 we describe the model, we give some basic notions on the random environment and present the main results. In Section 3 we give the properties of the random environment needed in Section 4 to prove the main results. In the Appendix we make the proof of the properties of the random environment.

2. DESCRIPTION OF THE MODEL AND MAIN RESULTS

2.1. Sinai's Random Walk Definition

Let $\alpha \equiv (\alpha_i, i \in \mathbb{Z})$ be a sequence of i.i.d. random variables taking values in $(0,1)$ defined on the probability space $(\Omega_1, \mathcal{F}_1, Q)$, this sequence will be called random environment. A random walk in random environment (denoted R.W.R.E.) $(X_n, n \in \mathbb{N})$ is a sequence of random variable taking value in $\mathbb{Z}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- for every fixed environment α, $(X_n, n \in \mathbb{N})$ is a Markov chain with the following transition probabilities, for all $i \in \mathbb{Z}$

$$\begin{aligned} &\mathbb{P}^{\alpha}[X_n = i+1 | X_{n-1} = i] = \alpha_i, \\ &\mathbb{P}^{\alpha}[X_n = i-1 | X_{n-1} = i] = 1 - \alpha_i \equiv \beta_i. \end{aligned} \tag{2.1}$$

We denote by $(\Omega_2, \mathcal{F}_2, \mathbb{P}^{\alpha})$ the probability space associated to this Markov chain.

- $\Omega = \Omega_1 \times \Omega_2$, $\forall A_1 \in \mathcal{F}_1$ and $\forall A_2 \in \mathcal{F}_2$, $\mathbb{P}[A_1 \times A_2] = \int_{A_1} Q(dw_1) \int_{A_2} \mathbb{P}^{\alpha(w_1)}(dw_2)$.

The probability measure $\mathbb{P}^{\alpha}[.|\, X_0 = a]$ will be denoted $\mathbb{P}_a^{\alpha}[.]$, the expectation associated to $\mathbb{P}_a^{\alpha}$: $\mathbb{E}_a^{\alpha}$, and the expectation associated to Q: $\mathbb{E}_Q$.

Now we introduce the hypothesis we use in all this work. Denoting $(\epsilon_i = \log[(1-\alpha_i)/\alpha_i], i \in \mathbb{Z})$, the two following hypothesis are the necessaries hypothesis

$$\mathbb{E}_Q[\epsilon_0] = 0, \tag{2.2}$$

$$\mathbb{E}_Q\left[\epsilon_0^2\right] \equiv \sigma^2 > 0. \tag{2.3}$$

Solomon[1] shows that under (2.2) the process $(X_n, n \in \mathbb{N})$ is $\mathbb{P}$ almost surely null recurrent and (2.3) implies that the model is not reduced to the simple random walk. In addition to (2.2) and (2.3) we will consider the following hypothesis of regularity, there exists $\kappa^+ \in \mathbb{R}_+^*$ such that for all $\kappa \in]0, \kappa^+[$

$$\mathbb{E}_Q[e^{\kappa \epsilon_0}] < \infty \text{ and } \mathbb{E}_Q\left[e^{-\kappa \epsilon_0}\right] < \infty. \tag{2.4}$$

We call *Sinai's random walk* the random walk in random environment previously defined with the three hypothesis (2.2), (2.3) and (2.4).

Notice that Y. Sinai used the stronger hypothesis:

$$\alpha_0 \geqslant \text{const} > 0, \;\; 1-\alpha_0 \geqslant \text{const} > 0. \tag{2.5}$$

The random potential and the valleys

Definition 2.1. The random potential $(S_k,\ k \in \mathbb{R})$ associated to the random environment α is defined by

$$S_k = \begin{cases} \sum_{1 \leqslant i \leqslant k} \epsilon_i, & k = 1, 2, \dots, \\ \sum_{k \leqslant i \leqslant -1} \epsilon_i, & k = -1, -2, \dots, \end{cases}$$

for the other $k \in \mathbb{R} \smallsetminus \mathbb{Z}$, (S_k, k) is defined by linear interpolation, and $S_0 = 0$. We denote $(S_t^n, t \in \mathbb{R})$ the normalized potential associated to $(S_k,\ k \in \mathbb{R})$

$$S_k^n = \frac{S_k}{\log n}, \quad k \in \mathbb{R}. \tag{2.6}$$

Definition 2.2. We will say that the triplet $\{\tilde{M}', \tilde{m}, \tilde{M}''\}$ is a *valley* if

$$S_{\tilde{M}'}^n = \max_{\tilde{M}' \leqslant t \leqslant \tilde{m}} S_t^n, \tag{2.7}$$

$$S_{\tilde{M}''}^n = \max_{\tilde{m} \leqslant t \leqslant \tilde{M}''} S_t^n, \tag{2.8}$$

$$S_{\tilde{m}}^n = \min_{\tilde{M}' \leqslant t \leqslant \tilde{M}''} S_t^n \; . \tag{2.9}$$

If $\tilde{m}$ is not unique, we choose the one with the smallest absolute value.

Definition 2.3. We will call *depth of the valley* $\{\tilde{M}', \tilde{m}, \tilde{M}''\}$ and we will denote $d([\tilde{M}', \tilde{M}''])$ the quantity

$$\min(S_{\tilde{M}'}^n - S_{\tilde{m}}^n, S_{\tilde{M}''}^n - S_{\tilde{m}}^n). \tag{2.10}$$

Now we define the operation of refinement.

Definition 2.4. Let $\{\tilde{M}', \tilde{m}, \tilde{M}''\}$ be a valley. Let $\tilde{M}_1$ and $\tilde{m}_1$ be such that $\tilde{m} \leqslant \tilde{M}_1 < \tilde{m}_1 \leqslant \tilde{M}''$ and

$$S_{\tilde{M}_1}^n - S_{\tilde{m}_1}^n = \max_{\tilde{m} \leqslant t' \leqslant t'' \leqslant \tilde{M}''} (S_{t'}^n - S_{t''}^n). \tag{2.11}$$

We say that the couple $(\tilde{M}_1, \tilde{m}_1)$ is obtained by a *right refinement* of $\{\tilde{M}', \tilde{m}, \tilde{M}''\}$. If the couple $(\tilde{m}_1, \tilde{M}_1)$ is not unique, we will take the ones such that $\tilde{m}_1$ and $\tilde{M}_1$ have the smallest absolute value. In a similar way, we define the left refinement operation.

In all this work, we denote $\log_p$ with $p \geqslant 2$ the p iterated logarithm and we assume that n is large enough such that $\log_p n$ is positive. Let $\gamma > 0$ a free parameter, denoting $\gamma(n) = (\gamma \log_2 n)(\log n)^{-1}$ we define what we will call a *valley containing* 0 *and of depth larger than* $1 + \gamma(n)$.

Definition 2.5. For $\gamma > 0$ and $n > 3$, we say that a valley $\{\tilde{M}', \tilde{m}, \tilde{M}''\}$ contains 0 and is of depth larger than $1 + \gamma(n)$ if and only if

1. $0 \in [\tilde{M}', \tilde{M}'']$,
2. $d\left(\{\tilde{M}', \tilde{M}''\}\right) \geqslant 1 + \gamma(n)$,
3. if $\tilde{m} < 0$, $S^n_{\tilde{M}''} - \max_{\tilde{m} \leqslant t \leqslant 0}\left(S^n_t\right) \geqslant \gamma(n)$,
   if $\tilde{m} > 0$, $S^n_{\tilde{M}'} - \max_{0 \leqslant t \leqslant \tilde{m}}\left(S^n_t\right) \geqslant \gamma(n)$.

The basic valley $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$

We recall the notion of *basic valley*, introduced by Y. Sinai and denoted here $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$. The definition we give is inspired by the work of Kesten.[6] First, let $\{\tilde{M}', \tilde{m}_0, \tilde{M}''\}$ be the smallest valley that contains 0 and of depth larger than $1 + \gamma(n)$. Here smallest means that if we construct, with the operation of refinement, other valleys in $\{\tilde{M}', \tilde{m}_0, \tilde{M}''\}$ such valleys will not satisfy one of the properties of Definition 2.5. $\tilde{M}'_0$ and $\tilde{M}_0$ are defined from $\tilde{m}_0$ in the following way
if $\tilde{m}_0 > 0$

$$\tilde{M}'_0 = \sup\left\{l \in \mathbb{Z}_-,\ l < \tilde{m}_0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n),\ S^n_l - \max_{0 \leqslant k \leqslant \tilde{m}_0} S^n_k \geqslant \gamma(n)\right\}, \tag{2.12}$$

$$\tilde{M}_0 = \inf\left\{l \in \mathbb{Z}_+,\ l > \tilde{m}_0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n)\right\}. \tag{2.13}$$

If $\tilde{m}_0 < 0$

$$\tilde{M}'_0 = \sup\left\{l \in \mathbb{Z}_-,\ l < \tilde{m}_0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n)\right\}, \tag{2.14}$$

$$\tilde{M}_0 = \inf\left\{l \in \mathbb{Z}_+,\ l > \tilde{m}_0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n),\ S^n_l - \max_{\tilde{m}_0 \leqslant k \leqslant 0} S^n_k \geqslant \gamma(n)\right\}. \tag{2.15}$$

If $\tilde{m}_0 = 0$

$$\tilde{M}'_0 = \sup\left\{l \in \mathbb{Z}_-,\ l < 0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n)\right\}, \tag{2.16}$$

$$\tilde{M}_0 = \inf\left\{l \in \mathbb{Z}_+,\ l > 0,\ S^n_l - S^n_{\tilde{m}_0} \geqslant 1 + \gamma(n)\right\}. \tag{2.17}$$

One can ask himself if the basic valley exists, in the Appendix A we prove the following lemma:

Lemma 2.6. Assume (2.2), (2.3) and (2.4), for all $\gamma > 0$ there exists $n_0 \equiv n_0(\gamma, \sigma, E[|\epsilon_0|^3])$ such that for all $n > n_0$

$$Q\left[\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\} \neq \varnothing\right] \geqslant 1 - (6\gamma \log_2 n)(\log n)^{-1}. \tag{2.18}$$

Remark 2.7. In all this paper we use the same notation n_0 for an integer that could change from line to line. Moreover, in the rest of the paper we do not always make explicit the dependance on γ of all those n_0 even if Lemma 2.6 is constantly used.

2.2. Main Results: Localization Phenomena

The following result shows that Sinai's random walk is sub-diffusive:

Proposition 2.8. There exists a strictly positive numerical constant $h > 0$, such that if (2.2) and (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) hold, for all $\gamma > 2$ there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$, there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and

$$\sup_{\alpha \in G_n}\left\{\mathbb{P}^\alpha_0\left[\bigcup_{m=0}^{n}\left\{X_m \notin \left[\tilde{M}'_0, \tilde{M}_0\right]\right\}\right]\right\} \leqslant \frac{2\log_2 n}{\sigma^2(\log n)^{\gamma-2}}, \tag{2.19}$$

moreover

$$\sup_{\alpha \in G_n}\left\{\mathbb{P}^\alpha_0\left[\bigcup_{m=0}^{n}\left\{X_m \notin \left[-(\sigma^{-1}\log n)^2 \log_2 n, (\sigma^{-1}\log n)^2 \log_2 n\right]\right\}\right]\right\} \leqslant \frac{2\log_2 n}{\sigma^2(\log n)^{\gamma-2}}. \tag{2.20}$$

Remark 2.9. A weaker form of this result can be found in the paper of Sinai (Lemma 3 p. 261).[3] The set G_n is called set of "good" environments. We will define it precisely in Section 3. This set is defined by collecting all the properties on the environment we need to prove our results.

Equation (2.19) shows that Sinai's walk is trapped in the basic valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ which is random, depending only on the random media and on n. More precisely, using (2.20), with an overwhelming probability $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ is within an interval centered at the origin and of size $2(\sigma^{-1}\log n)^2 \log_2 n$. In all this work h is a strictly positive numerical constant that can grow from line to line if needed.

The following remarkable result was proved by Sinai.[3]

Theorem 2.10. Assume (2.2), (2.3) and (2.5), for all $\epsilon > 0$ and all $\delta > 0$ there exists $n_0 \equiv n_0(\epsilon, \delta)$ such that for all $n > n_0$, there exists $C_n \subset \Omega_1$ with $Q[C_n] \geqslant 1 - \epsilon$ and

$$\lim_{n \to +\infty} \sup_{\alpha \in G_n} \mathbb{P}_0^\alpha \left[\left| \frac{X_n}{\log^2 n} - m_0 \right| > \delta \right] = 0, \tag{2.21}$$

$m_0 = \tilde{m}_0 (\log n)^{-2}$.

In this paper we improve Sinai's result in the following way, for all $\kappa \in]0, \kappa^+[$ we denote $\gamma_0 = \frac{12}{\kappa} + \frac{21}{2}$,

Theorem 2.11. There exists a strictly positive numerical constant $h > 0$, such that if (2.2) and (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) hold, for all $\gamma > \gamma_0$ there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$, there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and

$$\sup_{\alpha \in G_n} \left\{ \mathbb{P}_0^\alpha \left[\left| \frac{X_n}{\log^2 n} - m_0 \right| > \mathcal{G}\gamma \frac{(\log_2 n)^{9/2}}{(\log n)^{1/2}} \right] \right\} \leqslant \frac{4(\log_2 n)^{9/2}}{\sigma^{10}(\gamma \log n)^{\gamma - \gamma_0}}, \tag{2.22}$$

$m_0 = \tilde{m}_0 (\log n)^{-2}$ and $\mathcal{G} = (1600)^2$.

Remark 2.12. This result shows that, for a given instant n sufficiently large, with a Q probability tending to one, X_n belongs to a neighborhood of the point $\tilde{m}_0$ with a $\mathbb{P}^\alpha$ probability tending to one. The size of this neighborhood is of order $(\log n)^{3/2}(\log_2 n)^{9/2}$ that is negligible comparing to the typical range of Sinai's walk of order $(\log n)^2$. Moreover, an estimate on the rates of the convergence of these probabilities are given but we did not try any attempts to optimize these rates. However, if we look for an annealed result, that means a result in $\mathbb{P}$ probability, we get

$$\mathbb{P}\left[\left|\frac{X_n}{\log^2 n}-m_0\right|>\mathcal{G}\gamma\frac{(\log_2 n)^{9/2}}{(\log n)^{1/2}}\right]\leqslant 2h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} \tag{2.23}$$

and the rate in $(\log_3 n)(\log_2 n)^{-1}$ cannot be improved to something like $(\log n)^{-a}$ with $a>0$ without changing the size of the localization neighborhood.

We recall that the explicit limit distribution of m_0 was given independently by Kesten[(6)] and Golosov.[(5)]

2.3. Ideas of the Proofs

In this section we describe in detail the structure of the paper and give the main ideas of the proofs of Propositions 2.8 and Theorem 2.11. For these proofs we need both *arguments on the random environment* and *arguments on the random walk*.

Because of the technical aspect of the arguments on the environment, we summarize the needed *results on the environment* in Section 3 and we have put the proofs of these results in the Appendix at the end of the paper. So assuming the results of Section 3, the proofs of the main results are limited to the arguments for the walk given in Section 4.

Results on the random environment (Section 3). First we describe the ordered chopping in valleys. According to this construction, based on the refinement operation, we get a set of valleys with the two following main properties: 1. The valleys of this set are ordered (in the sense of the depth); 2. The depth of these valleys decrease when they get close to $\tilde{m}_0$. This construction is one of the important point to get estimations more precise than Sinai's ones, for the environment, and therefore for the walk. We have collected all the needed properties of the valleys in a definition (Definition 3.4). All the environments that satisfy this definition are called *good environment* and we get the set of good environment (called G_n, n is the time). The longest part of this work will be to prove that $Q[G_n]$ satisfies the mentioned estimate, this is the purpose of the Appendix.

Arguments for the walk (Section 4). First, we recall basic results on birth and death processes used all over the different proofs. We will always assume that the random environments belong to the set of good environments.

The proof of Proposition 2.8 is based on a basic argument: with an overwhelming probability, first the walk reach the bottom of the basic valley $\tilde{m}_0$ and then prefer returning n times to this point instead of climbing until the top of the valley (i.e. reaching one of the points $\tilde{M}_0'$ or $\tilde{M}_0$).

Moreover, according to one of the properties of the good environments, the size of the basic valley $max\{|\tilde{M}_0'|, |\tilde{M}_0|\} \leqslant (\sigma^{-1}\log n)^2 \log_2 n$. So we get the Proposition. We will see that to get this result we have used very few properties of the good environments.

The proof of Theorem 2.11 is based on the two following facts: **Fact 1**: With an overwhelming probability, the last return to $\tilde{m}_0$ before the instant n, occurs at an instant larger than $n - q_n$. q_n is a function of n given by $\log q_n \approx ((\log n)^{3/2}(\log_2 n)^{7/2})^{1/2}$. **Fact 2**: We use the same argument of the proof of Proposition 2.8. With an overwhelming probability, starting from $\tilde{m}_0$ with an amount of time $n-(n-q_n)=q_n$ the walk is trapped in a valley of size of order $(\log q_n)^2 \log_2 q_n \approx (\log n)^{3/2}(\log_2 n)^{9/2}$. This gives the Theorem.

The hardest part is to prove Fact 1, for this we use both an analysis of the return time to $\tilde{m}_0$ (Section 4.3) and the ordered chopping in valleys. The main idea is to prove that for each scale of time larger than q_n, the walk will return to $\tilde{m}_0$ with an overwhelming probability. These scales of time are chosen as function of the depth of the ordered valleys, i.e. for each scale of time corresponds a valleys. What we prove is that for each scale of time the walk can't be trapped in the corresponding valley. Indeed, starting from $\tilde{m}_0$, if the walk has enough time to reach the bottom of a valley it has enough time to escape from it and therefore to return to $\tilde{m}_0$.

Arguments for the random environment (Appendix). While the proof of the results for the random environment are technical we give some details. This provide completeness to the present paper and shows the difficulties to work with the hypothesis 2.4.

3. GOOD PROPERTIES OF A RANDOM ENVIRONMENT

In this section, we present different notions for the environment that are used to prove the main results. We give a method to classify some valleys obtained from $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ by the operation of refinement. To do this we need some basic result on $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$. Then we define the set of the "good" environments, this set contains all the environments that satisfy the needed properties to prove the main results.

3.1. Ordered Chopping in Valleys

Proposition 3.1. There exists $h>0$ such that if 2.2, 2.3 and 2.4 hold, for all $\gamma>0$ there exists $n_0 \equiv n_0(\gamma)$ such that for all $n>n_0$, we have

$$Q\left[\tilde{M}_0 \leqslant (\sigma^{-1}\log n)^2 \log_2 n\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}, \quad (3.1)$$

$$Q\left[\tilde{M}_0' \geqslant -(\sigma^{-1}\log n)^2\log_2 n\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}. \quad (3.2)$$

Before making a classification of the valleys we need to introduce the following notations, let $\gamma > 0$ and $n > 3$

$$b_n = [(\gamma)^{1/2}(\log n \log_2 n)^{3/2}], \quad (3.3)$$

$$k_n = ((\sigma^{-1}\log n)^2 \log_2 n)(b_n)^{-1}, \quad (3.4)$$

where $[a]$ is the integer part of $a \in \mathbb{R}$. Using 3.3 and 3.4 we construct a deterministic chopping of the interval $(-(\sigma^{-1}\log n)^2\log_2 n, (\sigma^{-1}\log n)^2\log_2 n)$ into pieces of length b_n. Moreover, we define:

$$l_n = D\sigma^2 \log k_n, \quad D = 1000. \quad (3.5)$$

We make the following construction, let us take $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ as the initial valley (see Section 2.1). Let us denote $\mathcal{M}_0' = \{\tilde{M}_0', \tilde{m}_0\}$ and $\mathcal{M}_0 = \{\tilde{m}_0, \tilde{M}_0\}$.

First, we consider the first *right refinement* of the valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ we denote $\{\tilde{M}_1, \tilde{m}_1\}$ the couple of maximizer and minimizer obtained after this refinement, let us add this points to the set $\mathcal{M}_0$ to get $\mathcal{M}_0 = \{\tilde{m}_0, \tilde{M}_1, \tilde{m}_1, \tilde{M}_0\}$. Now we consider the first refinement of $\{\tilde{m}_0, \tilde{M}_1\}$, we get the couple $\{\tilde{M}_2, \tilde{m}_2\}$ that we add to the set $\mathcal{M}_0$ and so on until we obtain the points $\{\tilde{M}_r, \tilde{m}_r\}$ such that $\tilde{M}_{r-1} - \tilde{m}_0 \geqslant l_n b_n$ and $\tilde{M}_r - \tilde{m}_0 \leqslant l_n b_n$. From this construction (see Fig. 1) we obtain a set of maximizer and minimizer (on the right of $\tilde{m}_0$) $\mathcal{M}_0 \equiv \left\{\tilde{m}_0, \tilde{M}_r, \tilde{m}_r, \ldots, \tilde{M}_1, \tilde{m}_1, \tilde{M}_0\right\}$.

In the same way we construct the set $\mathcal{M}_0'$ by making equivalent refinement on the left of the valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$. We make a first refinement that gives the points $\{\tilde{m}_1', \tilde{M}_1'\}$, then we refine $\{\tilde{M}_1', \tilde{m}_0\}$ and so on until we obtain $\{\tilde{m}_{r'}', \tilde{M}_{r'}'\}$ such that $\tilde{m}_0 - \tilde{M}_{r'-1}' \geqslant b_n l_n$ and $\tilde{m}_0 - \tilde{M}_{r'}' \leqslant b_n l_n$ (we denote $\mathcal{M}_0'$ this set of maximizer and minimizer on the left of $\tilde{m}_0$). Finally we get a set of maximizer and minimizer $\mathcal{M} \equiv \mathcal{M}_0' \cup \mathcal{M}_0 = \{\tilde{M}_0', \tilde{m}_1', \tilde{M}_1', \ldots, \tilde{M}_{r'}', \tilde{m}_0, \tilde{M}_r, \ldots, \tilde{M}_1, \tilde{m}_1, \tilde{M}_0\}$.

We will use the following notations,

$$\begin{array}{l|l} \text{If } 0 \leqslant i, j \leqslant r & \text{If } 0 \leqslant i, j \leqslant r' \\ \delta_{i,j} = S^n_{\tilde{M}_i} - S^n_{\tilde{m}_j}, & \delta_{i,j}' = S^n_{\tilde{M}_i'} - S^n_{\tilde{m}_j'}, \\ \eta_{i,j} = S^n_{\tilde{M}_i} - S^n_{\tilde{M}_j}, & \eta_{i,j}' = S^n_{\tilde{M}_i'} - S^n_{\tilde{M}_j'}, \\ \mu_{i,j} = S^n_{\tilde{m}_i} - S^n_{\tilde{m}_j}. & \mu_{i,j}' = S^n_{\tilde{m}_i'} - S^n_{\tilde{m}_j'}. \end{array} \quad (3.6)$$

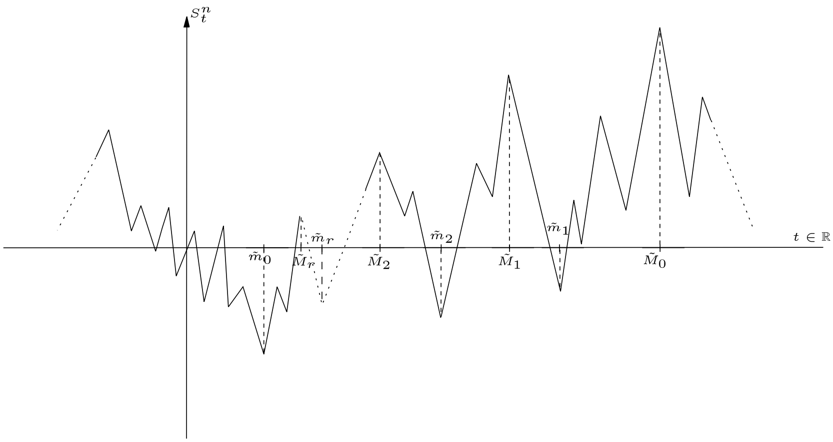

Fig. 1. Ordered chopping in valleys on the right hand side of $\tilde{m}_0$.

The beauty of the refinement is that we get immediately the following relations between the random variables defined in 3.6.

$$\delta_{0,0} > \delta_{1,1} > \cdots > \delta_{r,r} \geqslant 0, \tag{3.7}$$

$$\delta_{1,0} > \delta_{2,1} > \cdots > \delta_{r,0} \geqslant 0, \tag{3.8}$$

in the same way

$$\delta'_{0,0} > \delta'_{1,1} > \cdots > \delta'_{r,r} \geqslant 0, \tag{3.9}$$

$$\delta'_{1,0} > \delta'_{2,0} > \cdots > \delta'_{r',0} \geqslant 0, \tag{3.10}$$

and

$$\forall i,\ \ 0 \leqslant i \leqslant r-1,\ \ \eta_{i,i+1} \geqslant 0, \tag{3.11}$$

$$\forall i,\ \ 0 \leqslant i \leqslant r'-1,\ \ \eta'_{i,i+1} \geqslant 0. \tag{3.12}$$

We remark that the construction we made is possible if and only if $\tilde{m}_0 - \tilde{M}'_0 \geqslant b_n l_n$ and $\tilde{M}_0 - \tilde{m}_0 \geqslant l_n b_n$, but this is true with probability very near one, indeed the following lemma will be proved in the Appendix A.

Lemma 3.2. There exists $h > 0$ such that if (2.2), (2.3) and (2.4) hold, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\gamma)$ such that for all $n > n_0$, we have

$$Q\left[\tilde{M}_0 - \tilde{m}_0 \geqslant (\log n)^2 (65\sigma^2 \log_2 n)^{-1}\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2} \tag{3.13}$$

$$Q\left[\tilde{m}_0 - \tilde{M}_0' \geqslant (\log n)^2 (65\sigma^2 \log_2 n)^{-1}\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}. \tag{3.14}$$

3.2. Definition of the Set of Good Environments

Before defining a good environment, we introduce the following random variables, let $\gamma > 0$ and $n > 3$,

$$\tilde{M}_< = \sup\left\{m \in \mathbb{Z},\ m < \tilde{m}_0,\ S_m^n - S_{\tilde{m}_0}^n \geqslant \left(\log(q_n (\log n)^\gamma)\right)(\log n)^{-1}\right\}, \tag{3.15}$$
$$\tilde{M}_> = \inf\left\{m \in \mathbb{Z},\ m > \tilde{m}_0,\ S_m^n - S_{\tilde{m}_0}^n \geqslant \left(\log(q_n (\log n)^\gamma)\right)(\log n)^{-1}\right\},$$

where $q_n = \exp\left\{\left((200\sigma)^2 \gamma (\log_2 n)^{7/2} (\log n)^{3/2}\right)^{1/2}\right\}$.

Remark 3.3. Proposition 2.8 shows that for the scale of time n, Sinai's walk is trapped in the basic valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$. In the same way, we will prove that starting from $\tilde{m}_0$ with a scale of time q_n, Sinai's walk is trapped in the valley $\{\tilde{M}_<, \tilde{m}_0, \tilde{M}_>\}$. This argument will be used in the proof of Theorem 2.11.

Now we can define what we call a *good environment.*

Definition 3.4. Let $n > 3$, $\kappa \in]0, k_+[$, $\gamma > 0$ and $\omega \in \Omega_1$, we will say that $\alpha \equiv \alpha(\omega)$ is a *good environment* if the sequence $(\alpha_i,\ i \in \mathbb{Z}) \equiv (\alpha_i(\omega),\ i \in \mathbb{Z})$ satisfies the properties (3.16)–(3.36)

- The valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ exists: (3.16)

  $0 \in [\tilde{M}_0', \tilde{M}_0]$, (3.17)

  $\delta_{0,0} \geqslant 1 + \gamma(n),\ \ \delta_{0,0}' \geqslant 1 + \gamma(n)$, (3.18)

  If $\tilde{m}_0 > 0$, $S_{\tilde{M}_0'} - \max_{0 \leqslant m \leqslant \tilde{m}_0} \left(S_m^n\right) \geqslant \gamma(n)$, (3.19)

  if $\tilde{m}_0 < 0$, $S_{\tilde{M}_0} - \max_{\tilde{m}_0 \leqslant m \leqslant 0} \left(S_m^n\right) \geqslant \gamma(n)$. (3.20)

- $\max_{\tilde{M}_0' \leqslant l \leqslant \tilde{M}_0} \left((\alpha_l)^{-1}\right) \leqslant (\log n)^{\frac{6}{\kappa}}$, (3.21)

  $\max_{\tilde{M}_0' \leqslant l \leqslant \tilde{M}_0} \left((\beta_l)^{-1}\right) \leqslant (\log n)^{\frac{6}{\kappa}}$. (3.22)

- $\tilde{M}_0 \leqslant (\sigma^{-1} \log n)^2 \log_2 n,\ -\tilde{M}_0' \leqslant (\sigma^{-1} \log n)^2 \log_2 n$. (3.23)
- $\tilde{M}_< \geqslant \tilde{m}_0 - L_n, \quad \tilde{M}_> \leqslant \tilde{m}_0 + L_n$. (3.24)
- $r \leqslant 2(\log n)^{1/2} (\gamma \log_2 n)^{-1/2}$, (3.25)

$$r' \leqslant 2(\log n)^{1/2}(\gamma \log_2 n)^{-1/2}. \tag{3.26}$$

- For all $0 \leqslant i \leqslant r-1$

$$\eta_{i,i+1} \geqslant \gamma(n), \tag{3.27}$$
$$\delta_{i+1,i+1} \geqslant \gamma(n), \tag{3.28}$$
$$\mu_{i+1,0} \geqslant \gamma(n). \tag{3.29}$$

- For all $0 \leqslant i \leqslant r'-1$

$$\eta'_{i,i+1} \geqslant \gamma(n), \tag{3.30}$$
$$\delta'_{i+1,i+1} \geqslant \gamma(n), \tag{3.31}$$
$$\mu'_{i+1,0} \geqslant \gamma(n). \tag{3.32}$$

- $$\delta_{1,1} \leqslant 1-\gamma(n), \tag{3.33}$$
$$\delta'_{1,1} \leqslant 1-\gamma(n). \tag{3.34}$$

- $$\delta_{r,r} \leqslant (\log q_n)(\log n)^{-1}, \tag{3.35}$$
$$\delta'_{r',r'} \leqslant (\log q_n)(\log n)^{-1}, \tag{3.36}$$

where $L_n = \left(8\log[(\log n)^{\gamma} q_n]\sigma^{-1}\right)^2 \log_2 n$ and recalling that $q_n = \exp\left\{\left((200\sigma)^2\gamma \ (\log_2 n)^{7/2}(\log n)^{3/2}\right)^{1/2}\right\}$, $\delta_{.,.}$, $\delta'_{.,.}$, $\eta_{.,.}$, $\eta'_{.,.}$, $\mu_{.,.}$ and $\mu'_{.,.}$ are given by 3.6 and $\gamma(n) = (\gamma \log_2 n)(\log n)^{-1}$.

We define the set of *good environments* G_n as

$$G_n = \left\{\omega \in \Omega_1,\ \alpha(\omega) \text{ is a “good” environment}\right\}. \tag{3.37}$$

Remark 3.5. We remark that a good environment α is such that the different random variables $\tilde{M}_0, \tilde{M}'_0, \tilde{m}_0,\ r, r', \delta_{.,.}, \delta'_{.,.}\ \mu_{.,.}$ and $\mu'_{.,.}$ that depends on α satisfy some properties in relation to deterministic parameters like n, γ, σ and κ.

The properties (3.16)–(3.20) concern the existence of the basic valley $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$ with his main properties.

The properties (3.21) and (3.22) are technical properties due to the hypothesis 2.4. There is no equivalent properties in Sinai's paper because the stronger hypothesis 2.5 is used.

Equation (3.23) (respectively, (3.24)) give an upper bound of the distance between $\tilde{M}'_0$ and $\tilde{M}_0$ (respectively $\tilde{M}_<$ and $\tilde{M}_>$) and the origin (respectively to the random point $\tilde{m}_0$).

The properties from (3.25) to (3.36) concern the properties of the valleys obtained by the ordered chopping of $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$ effectuated in the previous paragraph. We remark that (3.25) and (3.26) give a deterministic upper bound for the number of right (respectively left) refinement performed in the ordered chopping in valleys, these upper bounds depend on n. This n dependance that does not appear in Sinai's work comes from the fact that we perform a chopping in valleys in such a way that the successive valleys are nested and contain $\tilde{m}_0$. This is a basic ingredient to get a result stronger than Sinai's one for the random walk itself.

Proposition 3.6. There exists $h>0$ such that if 2.2, 2.3 hold and for all $\kappa\in]0,\kappa^+[$ 2.4 hold, for all $\gamma>0$, there exists $n_0\equiv n_0(\kappa,\gamma)$ such that for all $n>n_0$

$$Q\left[G_n\right]\geqslant 1-h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}. \tag{3.38}$$

Proof. The proof of this proposition is done in the Appendix A. In fact $n_0\equiv n_0(\kappa,\gamma,\sigma,\mathbb{E}\left[|\epsilon_0|^3\right],\mathbb{E}\left[\epsilon_0^4\right],C)$, where $C=\mathbb{E}_Q\left[e^{\kappa\epsilon_0}\right]\vee\mathbb{E}_Q\left[e^{-\kappa\epsilon_0}\right]$ but for simplicity we do not always make explicit the dependance on $\sigma,\kappa,\mathbb{E}\left[|\epsilon_0|^3\right],\mathbb{E}\left[\epsilon_0^4\right]$ and C of n_0. ∎

4. PROOF OF THE MAIN RESULTS (PROPOSITION 2.8 AND THEOREM 2.11)

4.1. Basic Results for Birth and Death Processes

For completeness we recall some results of Chung[43] on inhomogeneous discrete time birth and death processes, we will always assume that α is fixed (denoted $\alpha\in\Omega_1$ in this work).

Let x, a and b in $\mathbb{Z}$, $a\neq b$, suppose $X_0=a$, denote

$$T_b^a=\begin{cases}\inf\{k\in\mathbb{N}^*,\ \ X_k=b\},\\ +\infty,\ \text{if such a } k \text{ not exists.}\end{cases} \tag{4.1}$$

Assume $a<x<b$, the two following lemmata can be found in Chung (pp. 73–76),[43] their proof follow from the method of difference equations.

Lemma 4.1. For all $\alpha\in\Omega_1$, we have

$$\mathbb{P}_x^{\alpha}\left[T_a^x>T_b^x\right]=\frac{\sum_{i=a+1}^{x-1}\exp\left(\log n\left(S_i^n-S_a^n\right)\right)+1}{\sum_{i=a+1}^{b-1}\exp\left(\log n\left(S_i^n-S_a^n\right)\right)+1}, \tag{4.2}$$

$$\mathbb{P}_x^{\alpha}\left[T_a^x<T_b^x\right]=\frac{\sum_{i=x+1}^{b-1}\exp\left(\log n\left(S_i^n-S_b^n\right)\right)+1}{\sum_{i=a+1}^{b-1}\exp\left(\log n\left(S_i^n-S_b^n\right)\right)+1}. \tag{4.3}$$

Let us denote $T_a^x \wedge T_b^x$ the minimum between T_a^x and T_b^x.

Lemma 4.2. For all $\alpha \in \Omega_1$, we have

$$\mathbb{E}_{a+1}^{\alpha}\left[T_a^{a+1} \wedge T_b^{a+1}\right] = \frac{\sum_{l=a+1}^{b-1}\sum_{j=l}^{b-1}\frac{1}{\alpha_l}F_n(j,l)}{\sum_{j=a+1}^{b-1}F_n(j,a)+1}, \tag{4.4}$$

$$\mathbb{E}_x^{\alpha}\left[T_a^x \wedge T_b^x\right] = \mathbb{E}_{a+1}^{\alpha}\left[T_a^{a+1} \wedge T_b^{a+1}\right]\left(1+\sum_{j=a+1}^{x-1}F_n(j,a)\right) - \sum_{l=a+1}^{x-1}\sum_{j=l}^{x-1}\frac{1}{\alpha_l}F_n(j,l), \tag{4.5}$$

where $F_n(j,l) = \exp\left(\log n\left(S_j^n - S_l^n\right)\right)$.

4.2. Proof of the Sub-diffusive Behavior (Proposition 2.8)

Ideas of the proof. First, we prove that starting from 0 the probability to hit $\tilde{m}_0$ before one of the points $\tilde{M}_0'-1$ or $\tilde{M}_0+1$ goes to 1 (Lemma 4.3) and starting from $\tilde{m}_0$ the probability of staying in the interval $[\tilde{M}_0', \tilde{M}_0]$ in a time n goes to 1 when n goes to infinity (Lemma 4.5).

In this section, we will always assume that $m_0 < 0$, (computations are the same for the other case).

Lemma 4.3. There exists $h > 0$ such that if (2.2) and (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 2$ there exists $n_0 \equiv n_0(\gamma, \kappa)$ such that for all $n > n_0$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and for all $\alpha \in G_n$

$$\mathbb{P}_0^{\alpha}\left[T_{\tilde{m}_0}^{\tilde{0}} \geqslant T_{\tilde{M}_0+1}^{\tilde{0}}\right] \leqslant \sigma^{-2}(\log_2 n)(\log n)^{-\gamma+2} + (n(\log n)^{\gamma})^{-1}. \tag{4.6}$$

Proof. Assume $\gamma > 2$, using Lemma 4.1 we easily get that

$$\mathbb{P}_0^{\alpha}\left[T_{\tilde{m}_0}^{\tilde{0}} \geqslant T_{\tilde{M}_0+1}^{\tilde{0}}\right] \leqslant |\tilde{m}_0| \max_{\tilde{m}_0+1 \leqslant i \leqslant -1}\left(\exp\left(-\log n\left(S_{\tilde{M}_0}^n - S_i^n\right)\right)\right) + 1$$

Using (3.20) and (3.23), we get (4.6). ∎

Remark 4.4. By hypothesis $\tilde{M}_0' < \tilde{m}_0 < 0$ therefore $\mathbb{P}^{\alpha}\left[T_{\tilde{m}_0}^{\tilde{0}} > T_{\tilde{M}_0'-1}^{\tilde{0}}\right] = 0.$

Lemma 4.5. There exists $h>0$ such that if (2.2) and (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 2$ there exists $n_0 \equiv n_0(\gamma, \kappa)$ such that for all $n > n_0$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ such that for all $\alpha \in G_n$ we have

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1} > n\right] \geqslant 1-(\log n)^{-\gamma}, \tag{4.7}$$

moreover

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{-[(\sigma^{-1}\log n)^2\log_2 n]-1} \wedge T^{\tilde{m}_0}_{[(\sigma^{-1}\log n)^2\log_2 n]+1} > n\right] \geqslant 1-(\log n)^{-\gamma}. \tag{4.8}$$

Proof. For all $i \geqslant 2$, define

$$T_i^{x\to x} = \begin{cases} \inf\{k > T_{i-1},\ \ X_t = x\}, \\ +\infty, \text{ if such } k \text{ does not exist.} \end{cases} \tag{4.9}$$

$$T_1^{x\to x} \equiv T^{x\to x} = \begin{cases} \inf\{k \in \mathbb{N}^*,\ \ X_k = x \text{ with } X_0 = x\}, \\ +\infty, \text{ if such } k \text{ does not exist.} \end{cases} \tag{4.10}$$

We denote $\tau_1 = T_1^{x\to x}$ and $\tau_i = T_i^{x\to x} - T_{i-1}^{x\to x}$, for all $i \geqslant 2$. Let $n \geqslant 1$, remark that $T_n^{\tilde{m}_0\to\tilde{m}_0} \equiv \sum_{i=1}^n \tau_i^{\tilde{m}_0\to\tilde{m}_0} > n$ so

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1} > n\right] = \mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1} > n, \sum_{i=1}^{n}\tau_i^{\tilde{m}_0\to\tilde{m}_0} > n\right] \tag{4.11}$$

$$\geqslant \mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1} > \sum_{i=1}^{n}\tau_i^{\tilde{m}_0\to\tilde{m}_0}\right]. \tag{4.12}$$

By the strong Markov property the random variables $(\tau_i, 1 \leqslant i \leqslant n)$ are i.i.d therefore

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1} > \sum_{i=1}^{n}\tau_i^{\tilde{m}_0\to\tilde{m}_0}\right] = \left(\mathbb{P}^{\alpha}\left[T^{\tilde{m}_0\to\tilde{m}_0} \leqslant T^{\tilde{m}_0}_{\tilde{M}'_0-1} \wedge T^{\tilde{m}_0}_{\tilde{M}_0+1}\right]\right)^n. \tag{4.13}$$

Moreover, it is easy to check that

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0\to\tilde{m}_0}\leqslant T^{\tilde{m}_0}_{\tilde{M}_0'-1}\wedge T^{\tilde{m}_0}_{\tilde{M}_0+1}\right]=\alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{M}_0+1}\leqslant T^{\tilde{m}_0+1}_{\tilde{m}_0}\right]$$
$$+\beta_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0-1}\left[T^{\tilde{m}_0-1}_{\tilde{M}_0'-1}\leqslant T^{\tilde{m}_0-1}_{\tilde{m}_0}\right]. \tag{4.14}$$

Using (4.2) and (3.18), we get that there exists $n_0\equiv n_0(\kappa,\gamma)$ such that for all $n>n_0$ and all $\alpha\in G_n$, $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{M}_0+1}< \ T^{\tilde{m}_0+1}_{\tilde{m}_0}\right]\leqslant n^{-(1+\gamma(n))}$, in the same way $\mathbb{P}^{\alpha}_{\tilde{m}_0-1}\left[T^{\tilde{m}_0-1}_{\tilde{M}_0'-1}<T^{\tilde{m}_0-1}_{\tilde{m}_0}\right]\leqslant n^{-(1+\gamma(n))}$. Using this and (4.14), we get for $n>n_0$ and all $\alpha\in G_n$

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}_0'-1}\wedge T^{\tilde{m}_0}_{\tilde{M}_0+1}<T^{\tilde{m}_0\to\tilde{m}_0}\right]\leqslant n^{-1-\gamma(n)}. \tag{4.15}$$

Replacing (4.15) in (4.13) and using (4.12) and the fact $(1-x)^n\geqslant 1-nx$, for all $0\leqslant x\leqslant 1$ and all $n\geqslant 1$ we get (4.7). For (4.8) we use (4.7) and (3.23). ∎

Proof (of Proposition 2.8). By the strong Markov property and Remark 4.4 we get that

$$\mathbb{P}^{\alpha}_{0}\left[\bigcap_{k=0}^{n}\left\{X_m\in\left[\tilde{M}_0',\tilde{M}_0\right]\right\}\right]\geqslant\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{M}_0'-1}\wedge T^{\tilde{m}_0}_{\tilde{M}_0+1}>n\right]-\mathbb{P}^{\alpha}_{0}\left[T^{\tilde{0}}_{\tilde{m}_0}>T^{\tilde{0}}_{\tilde{M}_0+1}\right]. \tag{4.16}$$

Using Lemmata 4.3 and 4.5, we get (2.19). We get (2.20) using (2.19) and (3.23). ∎

The next lemma will be used for the proof of Theorem 2.11.

Lemma 4.6. There exists $h>0$, such that if (2.2) and (2.3) hold and for all $\kappa\in]0,\kappa^+[$ (2.4) holds, for all $\gamma>2$ there exists $n_0\equiv n_0(\gamma,\kappa)$ such that for all $n>n_0$ there exists $G_n\subset\Omega_1$ with $Q\,[G_n]\geqslant 1-h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and for all $\alpha\in G_n$ we have

$$\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0}_{\tilde{m}_0-L_n}\wedge T^{\tilde{m}_0}_{\tilde{m}_0+L_n}>q_n\right]\geqslant 1-(\log n)^{-\gamma}, \tag{4.17}$$

where L_n and q_n are given at the end of Definition 3.4.

Proof. Using what we did to prove Lemma 4.5 replacing $\tilde{M}_0$ by $\tilde{M}_>$ and $\tilde{M}_0'$ by $\tilde{M}_<$ (see (3.15) for the definitions of $\tilde{M}_>$ and $\tilde{M}_>$), we easily get this lemma. ∎

4.3. Analysis of the Return Time $T^{\tilde m_0\to\tilde m_0}$

It is easy to check that $\mathbb{E}^{\alpha}_{\tilde m_0}\left[T^{\tilde m_0\to\tilde m_0}\right]=\infty$ $Q.a.s$, however, we will need an upper bound for the probability $\mathbb{P}^{\alpha}_{\tilde m_0}\left[T^{\tilde m_0\to\tilde m_0}>k\right]$ with $k>0$. We denote $a\vee b=\max(a,b)$.

Lemma 4.7. For all $\alpha\in\Omega_1$ and all $n>1$, we have for all i, $0\leqslant i\leqslant r$

$$\mathbb{E}^{\alpha}_{\tilde m_0+1}\left[\left(T^{\tilde m_0+1}_{\tilde m_0}\wedge T^{\tilde m_0+1}_{\tilde M_i+1}\right)^2\right]\leqslant D_i n^{(\delta_{i+1,i+1}-\eta_{i,i+1})\vee 0}, \tag{4.18}$$

with $D_i\equiv D_i(\alpha,n)=|\tilde M_i-\tilde m_0|^5\left(\max_{\tilde m_0\leqslant l\leqslant \tilde M_i}\left(\frac{1}{\alpha_l}\right)\right)^2$, and for all i, $0\leqslant i\leqslant r'$

$$\mathbb{E}^{\alpha}_{\tilde m_0-1}\left[\left(T^{\tilde m_0-1}_{\tilde m_0}\wedge T^{\tilde m_0-1}_{\tilde M'_i-1}\right)^2\right]\leqslant D'_i n^{(\delta'_{i+1,i+1}-\eta'_{i,i+1})\vee 0}, \tag{4.19}$$

with $D'_i\equiv D'_i(\alpha,n)=|\tilde M'_i-\tilde m_0|^5\left(\max_{\tilde M'_i\leqslant l\leqslant \tilde m_0}\left(\frac{1}{\beta_l}\right)\right)^2$. See 3.6 for the definitions of $\eta'_{i,i+1}$, $\delta'_{i+1,i+1}$, $\eta_{i,i+1}$ and $\delta_{i+1,i+1}$, recalling that r and r' are (respectively) the number of right (respectively left) refinement (see Section 3.1).

Proof. We only prove (4.18) (the proof of (4.19) is identical). It is easy to check, with the method of difference equations,

$$\mathbb{E}^{\alpha}_{\tilde m_0}\left[\left(T^{\tilde m_0}_{\tilde m_0+1}\wedge T^{\tilde m_0}_{\tilde M_i+1}\right)^2\right]=\frac{\sum_{l=\tilde m_0+1}^{\tilde M_i}\sum_{j=\tilde m_0+1}^{l}\frac{2u_l-1}{\alpha_l}F_n(j,l)}{\sum_{j=\tilde m_0+1}^{\tilde M_i}F_n(j,\tilde m_0)+1}, \tag{4.20}$$

with

$$u_l=\mathbb{E}^{\alpha}_{l}\left[T^{l}_{\tilde m_0}\wedge T^{l}_{\tilde M_i+1}\right], \tag{4.21}$$

u_l is given by (4.5) and $F_n(.,.)$ at the end of Lemma 4.2. First we give an upper bound of (4.21). Denoting $C_i\equiv C_i(\alpha,n)=\max_{\tilde m_0\leqslant l\leqslant \tilde M_i}\left(\frac{1}{\alpha_l}\right)(\tilde M_i-\tilde m_0)^2$ it is easy to check that $u_l\leqslant C_i\left(1+\sum_{j=\tilde m_0+1}^{l-1}F_n(j,\tilde m_0)\right)$. We have

$$\sum_{l=\tilde m_0+1}^{\tilde M_i}\sum_{j=\tilde m_0+1}^{l}\frac{2u_l-1}{\alpha_l}F_n(j,l)$$

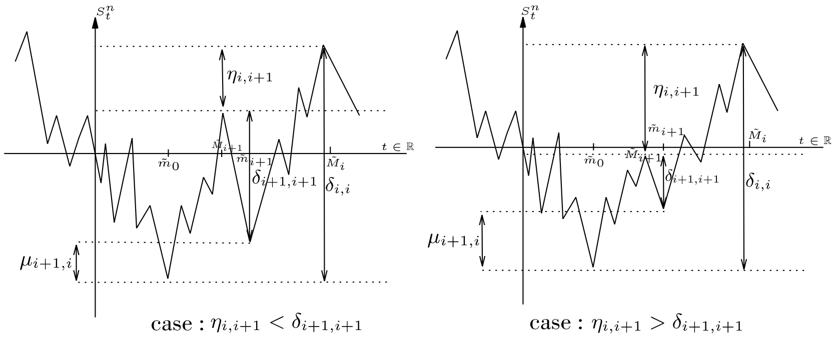

Fig. 2. Two possible trajectories of the random potential and the associated (right) refinements of $\{\tilde{m}_0, \tilde{M}_i\}$.

$$\leqslant 2C_i \sum_{l=\tilde{m}_0+1}^{\tilde{M}_i} \sum_{j=\tilde{m}_0+1}^{l} \left(1+\sum_{i=\tilde{m}_0+1}^{l-1} F_n(i,\tilde{m}_0)\right)(\alpha_l)^{-1} F_n(j,l). \tag{4.22}$$

Now let us consider the first refinement of $\{\tilde{m}_0, \tilde{M}_i\}$, denote $\tilde{m}_{i+1}$ the minimizer obtained and $\tilde{M}_{i+1}$ the maximizer, it is easy to check (see Fig. 2) that

$$\begin{aligned}\sum_{l=\tilde{m}_0+1}^{\tilde{M}_i} \sum_{j=\tilde{m}_0+1}^{l} & \frac{\left(1+\sum_{i=\tilde{m}_0+1}^{l-1} F_n(i,\tilde{m}_0)\right)}{\alpha_l} F_n(j,l) \\ & \leqslant \frac{|\tilde{M}_i-\tilde{m}_0|^3}{2} \max_{\tilde{m}_0 \leqslant l \leqslant \tilde{M}_i} \left(\frac{1}{\alpha_l}\right) n^{(\delta_{i,0})\vee(\delta_{i+1,0}+\delta_{i+1,i+1})},\end{aligned} \tag{4.23}$$

where $\delta_{.,.}$ is given in (3.6). Using (4.22) and (4.23) we get

$$\sum_{l=\tilde{m}_0+1}^{\tilde{M}_i} \sum_{j=\tilde{m}_0+1}^{l} \frac{2u_l-1}{\alpha_l} F_n(j,l) \leqslant D_i \times n^{(\delta_{i,0})\vee(\delta_{i+1,0}+\delta_{i+1,i+1})}, \tag{4.24}$$

where $D_i \equiv D_i(\alpha,n) = |\tilde{M}_i-\tilde{m}_0|^5 \left(\max_{\tilde{m}_0 \leqslant l \leqslant \tilde{M}_i}\left(\frac{1}{\alpha_l}\right)\right)^2$.

Moreover, it is easy to check that $\sum_{j=\tilde{m}_0+1}^{\tilde{M}_i} F_n(j,\tilde{m}_0) \geqslant n^{\delta_{i,0}}$, replacing this and (4.24) in (4.20) and noticing that $\delta_{i+1,0}-\delta_{i,0} = -\eta_{i,i+1}$ we get (4.18). ∎

Proposition 4.8. For all $\alpha \in \Omega_1$, $n > 1$ and $q > 0$ we have, for all i, $0 \leqslant i \leqslant r$

$$\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > q\right] \leqslant (D_i n^{(\delta_{i+1,i+1}-\eta_{i,i+1})\vee 0})q^{-2} + n^{-\delta_{i,0}}, \tag{4.25}$$

with $D_i = |\tilde{M}_i - \tilde{m}_0|^5 \left(\max_{\tilde{m}_0 \leqslant l \leqslant \tilde{M}_i}\left(\frac{1}{\alpha_l}\right)\right)^2$, and for all i, $0 \leqslant i \leqslant r'$

$$\mathbb{P}^{\alpha}_{\tilde{m}_0-1}\left[T^{\tilde{m}_0-1}_{\tilde{m}_0} > q\right] \leqslant (D'_i n^{(\delta'_{i+1,i+1}-\eta'_{i,i+1})\vee 0})q^{-2} + n^{-\delta'_{i,0}}, \tag{4.26}$$

with $D'_i = |\tilde{M}'_i - \tilde{m}_0|^5 \left(\max_{\tilde{M}'_i \leqslant l \leqslant \tilde{m}_0}\left(\frac{1}{\beta_l}\right)\right)^2$. See 3.6 for the definitions of $\eta'_{i,i+1}$, $\delta'_{i+1,i+1}$, $\eta_{i,i+1}$ and $\delta_{i+1,i+1}$, recalling that r and r' are (respectively) the number of right (respectively left) refinement (see Section (3.1)).

Remark 4.9. Equation (4.25) does not imply that $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > q\right]$ is sumable on q, indeed on the right hand side of 4.25, "$n^{-\delta_{i,0}}$" does not depend on q.

Proof (of Proposition 4.8). Let us estimate $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > q\right]$, let $0 \leqslant i \leqslant r$, we have

$$\begin{aligned}\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > q\right] \leqslant\ & \mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} \wedge T^{\tilde{m}_0+1}_{\tilde{M}_i+1} > q\right] \\ & + \mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > T^{\tilde{m}_0+1}_{\tilde{M}_i+1}\right].\end{aligned} \tag{4.27}$$

Using (4.3) and recalling that $\delta_{i,0} = S^n_{\tilde{M}_i} - S^n_{\tilde{m}_0}$ we get $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > T^{\tilde{m}_0+1}_{\tilde{M}_i+1}\right] \leqslant n^{-\delta_{i,0}}$. Moreover, by Markov inequality we have $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} \wedge T^{\tilde{m}_0+1}_{\tilde{M}_i+1} > q\right] \leqslant \left(\mathbb{E}^{\alpha}_{\tilde{m}_0+1}\left[\left(T^{\tilde{m}_0+1}_{\tilde{m}_0} \wedge T^{\tilde{m}_0+1}_{\tilde{M}_i+1}\right)^2\right]\right)q^{-2}$. To end the proof we use (4.18) (similar computations give (4.26)). ∎

4.4. Proof of Theorem 2.11

The sketch of the proof is the following we prove (with a probability very near one) that $(X_k)_{1\leqslant k\leqslant n}$ hit $\tilde{m}_0$ in a time smaller than n. Then we show that it does not exist an instant $1 \leqslant k \leqslant n - q_n$ ($q(n)$ is given at the end of Definition 3.4) such that the R.W.R.E. will not return to $\tilde{m}_0$ (Proposition 4.10). Finally, we prove that starting from $\tilde{m}_0$, in a time smaller than $n-(n-q_n) = q_n$ the R.W.R.E. cannot escape from a region which size is of order $(\log q_n)^2$ (Proposition 4.14).

First we introduce the next event, let $n>1$ and $1 \leqslant q \leqslant n$

$$\mathcal{A}_q = \bigcup_{n-q \leqslant k \leqslant n} \{X_k = \tilde{m}_0\}. \tag{4.28}$$

Let $\delta_q > 0$, we have

$$\mathbb{P}_0^\alpha\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_q\right] \leqslant \mathbb{P}_0^\alpha\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_q,\ \mathcal{A}_q\right] + \mathbb{P}_0^\alpha\left[\mathcal{A}_q^c\right]. \tag{4.29}$$

Now we estimate each probability of the right hand side of (4.29) in Propositions 4.10 and 4.14.

Proposition 4.10. There exists $h>0$ such that if (2.2) and (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds, for all $\gamma > 12/\kappa + 21/2$ there exists $n_0 \equiv n_0(\gamma,\kappa)$ such that for all $n > n_0$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)/(\log_2 n)\right)^{1/2}$ and for all $\alpha \in G_n$

$$\begin{aligned}\mathbb{P}_0^\alpha\left[\mathcal{A}_{q_n}^c\right] \leqslant\ & \frac{2(\log_2 n)^{9/2}}{(\gamma)^{1/2}(\log n)^{\gamma-(12/\kappa+21/2)}} \\ & + \mathcal{O}\left(\frac{(\log_2)^2}{(\log n)^{\gamma-(6/\kappa+4)}}\right),\end{aligned} \tag{4.30}$$

q_n is given at the end of Definition 3.4.

Proof. First we remark that for all $n>1$ and all $1 \leqslant q \leqslant n$

$$\mathbb{P}_0^\alpha\left[\mathcal{A}_q^c\right] \leqslant \mathbb{P}_0^\alpha\left[T_{\tilde{m}_0}^0 > n\right] + \mathbb{P}_0^\alpha\left[\mathcal{A}_q^c,\ T_{\tilde{m}_0}^0 \leqslant n\right]. \tag{4.31}$$

We estimate each term of the right hand side of 4.31, the first one in Lemma (4.11) and the second in Lemma 4.12. ∎

Lemma 4.11. There exists $h>0$ such that if (2.2) and (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds, for all $\gamma > \frac{6}{\kappa} + 4$, there exists $n_1' \equiv n_1'(\kappa,\gamma)$ such that for all $n > n_1'$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)/(\log_2 n)\right)^{1/2}$ and for all $\alpha \in G_n$, we have

$$\mathbb{P}_0^\alpha\left[T_{\tilde{m}_0}^0 > n\right] \leqslant \frac{5(\log_2 n)^2}{\sigma^4(\log n)^{\gamma-\left(\frac{6}{\kappa}+4\right)}}. \tag{4.32}$$

Proof. Let us consider the valley $\{\tilde{M}'_0, \tilde{m}_0, \tilde{M}_0\}$, we assume $\tilde{m}_0 > 0$ (computations are similar if $\tilde{m}_0 \leqslant 0$). We have

$$\mathbb{P}_0^\alpha\left[T^0_{\tilde{m}_0} > n\right] \leqslant \mathbb{P}_0^\alpha\left[T^0_{\tilde{m}_0} \wedge T^0_{\tilde{M}'_0-1} > n\right] + \mathbb{P}_0^\alpha\left[T^0_{\tilde{M}'_0-1} < T^0_{\tilde{m}_0}\right]. \tag{4.33}$$

For the second probability on the right hand side of (4.33) we have already see (Lemma 4.3) that for all $\gamma > 2$ there exists $n_1 \equiv n_1(\kappa, \gamma)$ such that for all $n > n_1$ and all $\alpha \in G_n$

$$\mathbb{P}_0^\alpha\left[T^0_{\tilde{M}'_0-1} < T^0_{\tilde{m}_0}\right] \leqslant \sigma^{-2}\log_2 n(\log n)^{-\gamma+2}. \tag{4.34}$$

For the first probability on the right hand side of (4.33) we have by the Markov inequality

$$\mathbb{P}_0^\alpha\left[T^0_{\tilde{m}_0} \wedge T^0_{\tilde{M}'_0-1} > n\right] \leqslant \mathbb{E}_0\left[T^0_{\tilde{m}_0} \wedge T^0_{\tilde{M}'_0-1}\right] n^{-1}. \tag{4.35}$$

To compute the mean in (4.35) we use Lemma 4.5, it is easy to check that:

$$\mathbb{E}_0^\alpha\left[T^0_{\tilde{M}'_0-1} \wedge T^0_{\tilde{m}_0}\right] \leqslant \sum_{l=\tilde{M}'_0}^{\tilde{m}_0-1}\sum_{j=l}^{\tilde{m}_0-1}\frac{1}{\alpha_l}F_n(j,l), \tag{4.36}$$

where $F_n(j,l) = \exp\left(\log n(S^n_l - S^n_j)\right)$. Let us consider the first refinement of $\{\tilde{M}'_0, \tilde{m}_0\}$, it gives the point $\tilde{M}'_1$ (for the maximizer) and $\tilde{m}'_1$ (for the minimizer), so we get

$$\sum_{l=\tilde{M}'_0}^{\tilde{m}_0-1}\sum_{j=l}^{\tilde{m}_0-1}\frac{1}{\alpha_l}F_n(j,l) \leqslant C_0 n^{\delta'_{1,1}}, \tag{4.37}$$

where $\delta'_{1,1} \equiv S^n_{\tilde{M}'_1} - S^n_{\tilde{m}'_1}$ and $C_0 \equiv C_0(\alpha, n) = (\tilde{M}'_0 - \tilde{m}_0)^2 \max_{\tilde{M}'_0 \leqslant l \leqslant \tilde{m}_0}\left(\frac{1}{\alpha_l}\right)$. Using (4.37), (4.36) and (4.35) we get

$$\mathbb{P}_0^\alpha\left[T^0_{\tilde{m}_0} \wedge T^0_{\tilde{M}'_0-1} > n\right] \leqslant (C_0 n^{\delta'_{1,1}})n^{-1}. \tag{4.38}$$

Using formulas (3.21), (3.23) and (3.34) we get that for all $\gamma > \frac{6}{\kappa} + 4$, there exists $n_2 \equiv n_2(\gamma)$ such that for all $n > n_2$ and $\alpha \in G_n$

$$\mathbb{P}_0^\alpha \left[T^0_{\tilde{m}_0} \wedge T^0_{\tilde{M}'_0} > n \right] \leqslant \frac{(2\log_2 n)^2}{\sigma^4 (\log n)^{\gamma - \left(\frac{6}{\kappa}+4\right)}}. \tag{4.39}$$

We get (4.32) using (4.33), (4.34) and (4.39) and taking $n'_1 = n_1 \vee n_2$. ∎

Lemma 4.12. There exists $h > 0$, such that if (2.2) and (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 12/\kappa + 21/2$ there exists $n_0 \equiv n_0(\gamma, \kappa)$ such that for all $n > n_0$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and for all $\alpha \in G_n$

$$\mathbb{P}_0^\alpha \left[\mathcal{A}^c_{q_n},\ T^0_{\tilde{m}_0} \leqslant n \right] \leqslant \frac{3(\log_2 n)^{9/2}}{\sigma^{10}(\gamma)^{1/2}(\log n)^{\gamma - (\frac{12}{\kappa}+\frac{21}{2})}} + \mathcal{O}\left(\frac{1}{(\log n)^{\gamma - 1/2}(\log_2 n)^{1/2}} \right) \tag{4.40}$$

q_n is given at the end of Definition 3.4.

Proof. We recall that for all $1 \leqslant q \leqslant n$ we have denoted $\mathcal{A}^c_q = \bigcap_{n-q \leqslant k \leqslant n} \{X_k \neq \tilde{m}_0\}$. Denoting

$$\bar{\mathcal{A}}^c_q = \bigcup_{1 \leqslant p \leqslant n-q-1} \left\{ \{X_p = \tilde{m}_0\} \bigcap_{m=p+1}^{n} \{X_m \neq \tilde{m}_0\} \right\}, \tag{4.41}$$

we remark that $\left\{\mathcal{A}^c_q, T^0_{\tilde{m}_0} \leqslant n\right\} \subset \bar{\mathcal{A}}^c_q$. Therefore, we only have to give an upper bound of $\mathbb{P}_0^\alpha\left[\bar{\mathcal{A}}^c_q\right]$, by the Markov property we have

$$\mathbb{P}_0^\alpha\left[\bar{\mathcal{A}}^c_q\right] = \sum_{1 \leqslant p \leqslant n-q-1} \mathbb{P}^\alpha_{\tilde{m}_0} \left[\bigcap_{m=1}^{n-p} \{X_m \neq \tilde{m}_0\} \right] \mathbb{P}_0^\alpha\left[X_p = \tilde{m}_0\right]. \tag{4.42}$$

Using the change $k = n - p$, we get

$$\begin{aligned} \mathbb{P}_0^\alpha\left[\bar{\mathcal{A}}^c_q\right] &\leqslant \sum_{q+1 \leqslant k \leqslant n-1} \mathbb{P}^\alpha_{\tilde{m}_0} \left[\bigcap_{m=1}^{k} \{X_m \neq \tilde{m}_0\} \right] \\ &\equiv \sum_{q+1 \leqslant k \leqslant n-1} \mathbb{P}^\alpha_{\tilde{m}_0} \left[T^{\tilde{m}_0 \to \tilde{m}_0} > k \right]. \quad ∎ \end{aligned} \tag{4.43}$$

Remark 4.13. We recall that R.W.R.E. is null recurrent $\mathbb{P}.a.s$, so for the moment, we can't say anything on $\sum_{q+1\leqslant k\leqslant n-1}\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0\to\tilde{m}_0}>k\right]$.

First, let us decompose the sum in (4.43)

$$\sum_{q+1\leqslant k\leqslant n-1}\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0\to\tilde{m}_0}>k\right]=\sum_{q\leqslant k\leqslant n-2}\alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right] \tag{4.44}$$

$$+\sum_{q\leqslant k\leqslant n-2}\beta_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0-1}\left[T^{\tilde{m}_0-1}_{\tilde{m}_0}>k\right]. \tag{4.45}$$

Let us give an upper bound to the sum on the right-hand side of (4.44). We want to find q as small as possible but such that this sum goes to 0. For this we use step by step the inequality (4.25) to $\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right]$: we have

$$\sum_{[n^{\delta_{r,r}}]+1\leqslant k\leqslant n-2}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right]=\sum_{k=[n^{\delta_{1,1}}]+1}^{n-2}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right] \tag{4.46}$$

$$+\sum_{i=1}^{r-1}\sum_{k=[n^{\delta_{i+1,i+1}}]+1}^{[n^{\delta_{i,i}}]}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right]. \tag{4.47}$$

For the sum on the right hand side of (4.46), by inequality (4.25) (taking $i=0$) we have

$$\sum_{k=[n^{\delta_{1,1}}]+1}^{n-2}\mathbb{P}^{\alpha}_{\tilde{m}_0}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right]\leqslant\frac{n-n^{\delta_{1,1}}}{n^{\delta_{0,0}}}+\sum_{k=[n^{\delta_{1,1}}]+1}^{n}\frac{D_0 n^{(\delta_{1,1}-\eta_{0,1})\vee 0}}{k^2} \tag{4.48}$$

$$\leqslant\frac{n}{n^{\delta_{0,0}}}+\frac{D_0}{n^{\delta_{1,1}\wedge\eta_{0,1}}}, \tag{4.49}$$

where $D_0=|\tilde{M}_0-\tilde{m}_0|^5\left(\max_{\tilde{m}_0\leqslant l\leqslant\tilde{M}_0}\left(\frac{1}{\alpha_l}\right)\right)^2$. For the other terms ($1\leqslant i\leqslant r-1$) of the sum in (4.47), using the inequality (4.25) we have

$$\sum_{k=[n^{\delta_{i+1,i+1}}]+1}^{[n^{\delta_{i,i}}]}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0}>k\right]\leqslant\frac{n^{\delta_{i,i}}-n^{\delta_{i+1,i+1}}}{n^{\delta_{i,0}}}$$

$$+\sum_{k=[n^{\delta_{i+1,i+1}}]+1}^{[n^{\delta_{i,i}}]}\frac{D_i(n^{(\delta_{i+1,i+1}-\eta_{i,i+1})\vee 0}}{k^2} \tag{4.50}$$

$$\leqslant \frac{1}{n^{\mu_{i,0}}}+\frac{D_i}{n^{\delta_{i+1,i+1}\wedge\eta_{i,i+1}}}, \tag{4.51}$$

where we have used that $\delta_{i,0}-\delta_{i,i}=\mu_{i,0}$ and $D_i=|\tilde{M}_i-\tilde{m}_0|^5\Big(\max_{\tilde{m}_0\leqslant l\leqslant\tilde{M}_i}\left(\frac{1}{\alpha_l}\right)\Big)^2$. So, for the sum (4.47) we get from (4.51) that

$$\sum_{i=1}^{r-1}\sum_{k=[n^{\delta_{i+1,i+1}}]+1}^{[n^{\delta_{i,i}}]}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T_{\tilde{m}_0}^{\tilde{m}_0+1}>k\right]\leqslant\sum_{i=1}^{r-1}\frac{1}{n^{\mu_{i,0}}}+\sum_{i=1}^{r-1}\frac{D_i}{n^{\delta_{i+1,i+1}\wedge\eta_{i,i+1}}} \tag{4.52}$$

$$\leqslant \frac{r-1}{n^{\min_{1\leqslant i\leqslant r-1}(\mu_{i,0})}}+\frac{(r-1)D_0}{n^{\min_{1\leqslant i\leqslant r-1}(\delta_{i+1,i+1}\wedge\eta_{i,i+1})}}, \tag{4.53}$$

and we have used that D_i is decreasing in i. Collecting the terms (4.53) and (4.49) we get

$$\sum_{[n^{\delta_{r,r}}]+1\leqslant k\leqslant n-2}\alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T_{\tilde{m}_0}^{\tilde{m}_0+1}>k\right]\leqslant\frac{n}{n^{\delta_{0,0}}}+\frac{r-1}{n^{\min_{1\leqslant i\leqslant r-1}(\mu_{i,0})}}+\frac{rD_0}{n^{\min_{0\leqslant i\leqslant r-1}(\delta_{i+1,i+1}\wedge\eta_{i,i+1})}}. \tag{4.54}$$

Now using the good properties (3.18), (3.21), (3.27)–(3.29), (3.23) and (3.25) we easily get that for all $\gamma>\frac{12}{\kappa}+\frac{21}{2}$, there exist n_1 such that for all $n>n_1$, $\alpha\in G_n$,

$$\sum_{[n^{\delta_{r,r}}]+1\leqslant k\leqslant n-2}\alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T_{\tilde{m}_0}^{\tilde{m}_0+1}>k-1\right]\leqslant\frac{3(\gamma\log_2 n)^{9/2}}{\sigma^{10}(\gamma)^{1/2}(\log n)^{\gamma-(\frac{12}{\kappa}+\frac{21}{2})}}. \tag{4.55}$$

Finally, using (3.35) and therefore choosing $q=[q_n]$, where q_n is given at the end of Definition 3.4, we get that for all $\gamma > \frac{12}{\kappa} + \frac{21}{2}$, $n > n_1$ and $\alpha \in G_n$

$$\sum_{q=[q_n]\leqslant k\leqslant n-2} \alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > k\right] \leqslant \sum_{q=[n^{\delta_{r,r}}]+1\leqslant k\leqslant n-2} \alpha_{\tilde{m}_0}\mathbb{P}^{\alpha}_{\tilde{m}_0+1}\left[T^{\tilde{m}_0+1}_{\tilde{m}_0} > k\right] \tag{4.56}$$

$$\leqslant \frac{3(\log_2 n)^{9/2}}{\sigma^{10}(\gamma)^{1/2}(\log n)^{\gamma-(\frac{12}{\kappa}+\frac{21}{2})}}. \tag{4.57}$$

Making similar computation for the sum on the right-hand side of (4.45) one get the same upper bound with $q=[q_n]$. Using these estimates, (4.45), (4.44), (4.43) and the fact $\left\{\mathcal{A}^c_q, T^0_{\tilde{m}_0} \leqslant n\right\} \subset \bar{\mathcal{A}}^c_q$ we get the lemma taking $q=[q_n]$ and $n''_1 = n_1$.

We get Proposition 4.10 collecting the results of Lemmata 4.11, 4.12, using 4.31 and taking $n'_0 = n'_1 \vee n''_1$ and $q=[q_n]$.

Proposition 4.14. There exists $h>0$, such that if (2.2) and (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\gamma,\kappa)$ such that for all $n > n_0$ there exists $G_n \subset \Omega_1$ with $Q[G_n] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}$ and for all $\alpha \in G_n$

$$\mathbb{P}^{\alpha}_0\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_{q_n},\ \mathcal{A}_{q_n}\right] \leqslant \frac{1}{(\log n)^{\gamma}}, \tag{4.58}$$

$\delta_{q_n} = L_n(\log n)^{-2}$, q_n and L_n are given at the of Definition 3.4.

Proof. Let us introduce the following stopping time $T_{\tilde{m}_0}(q) = \inf\{l \geqslant n-q,\ X_l = \tilde{m}_0\}$. We remark that $\mathcal{A}_q \Leftrightarrow n-q \leqslant T_{\tilde{m}_0}(q) \leqslant n$. Taking $q=[q_n]$, by the strong Markov property we have

$$\mathbb{P}^{\alpha}_0\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_{q_n},\ \mathcal{A}_{[q_n]}\right] = \sum_{l=n-[q_n]}^{n} \mathbb{P}^{\alpha}_{\tilde{m}_0}\left[\left|\frac{X_{n-l}}{(\log n)^2} - m_0\right| > \delta_{q_n}\right]\mathbb{P}^{\alpha}_0\left[T_{\tilde{m}_0}(q_n) = l\right]. \tag{4.59}$$

Therefore we get

$$\mathbb{P}^{\alpha}_0\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_{q_n},\ \mathcal{A}_{[q_n]}\right]$$

$$\leqslant \sum_{l=0}^{q_n} \mathbb{P}_{\tilde{m}_0}^{\alpha}\left[T_{\tilde{m}_0+L_n}^{\tilde{m}_0} \wedge T_{\tilde{m}_0-L_n}^{\tilde{m}_0} < q_n - l\right] \mathbb{P}_0^{\alpha}\left[T_{\tilde{m}_0}(q_n) = l\right] \tag{4.60}$$

$$\leqslant \mathbb{P}_{\tilde{m}_0}^{\alpha}\left[T_{\tilde{m}_0+L_n}^{\tilde{m}_0} \wedge T_{\tilde{m}_0-L_n}^{\tilde{m}_0} < q_n\right], \tag{4.61}$$

Using Lemma 4.6 we get (4.58).

Now we end the proof of theorem 2.11.

Assume (2.2), (2.3) hold, let $\kappa \in]0, \kappa^+[$ such that (2.4) hold, let us denote $\gamma_0 = \frac{12}{\kappa} + \frac{21}{2}$, let $\gamma > \gamma_0$. Taking $q = [q_n]$ and $\delta_q = L_n (\log n)^{-2}$ in (4.29) we obtain from Propositions 4.10 and 4.14 that there exists $n_1 \equiv n_1(\kappa, \gamma)$ such that for all $n > n_1$ and all $\alpha \in G_n$

$$\mathbb{P}_0^{\alpha}\left[\left|\frac{X_n}{(\log n)^2} - m_0\right| > \delta_{q_n}\right] \leqslant \frac{3(\log_2 n)^{9/2}}{\sigma^{10}(\gamma)^{1/2}(\log n)^{\gamma-\gamma_0}} + \mathcal{O}\left(\frac{1}{(\log n)^{\gamma-(6/\kappa+4)}}\right), \tag{4.62}$$

Moreover we remark that one can find $n_2 > n_1$ such that for all $n > n_2$ we have $\delta_{q_n} \equiv L_n (\log n)^{-2} \leqslant \gamma (1600)^2 (\log_2 n)^{9/2} (\log n)^{-1/2}$. ∎

APPENDIX A. PROOF OF THE GOOD PROPERTIES FOR THE ENVIRONMENT (PROPOSITION 3.6)

In all this section we will use standard facts on sums of i.i.d. random variables, these results are summarized in the Section B of this appendix.

Elementary results on the basic valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$

We introduce the following stopping times, for $a > 0$,

$$U_a^+ \equiv U_a^+(S_j^n, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, \ \ S_m^n \geqslant a\}, \\ +\infty, \text{ if such a } m \text{ does not exist.} \end{cases} \tag{A.1}$$

$$U_a^- \equiv U_a^-(S_j^n, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*, \ \ S_m^n \leqslant -a\}, \\ +\infty, \text{ if such a } m \text{ does not exist.} \end{cases} \tag{A.2}$$

Proof of Lemma 2.6. To prove this lemma it is enough to prove that the valley $\{U_{1+\gamma(n)}^-, \tilde{m}, U_{1+\gamma(n)}^+\}$ satisfies the three properties of Definition 2.5 with a probability very near 1. Let $\kappa \in]0, \kappa^+[$, and $\gamma > 0$. By definition of $U_{1+\gamma(n)}^-$ and $U_{1+\gamma(n)}^+$, $\{U_{1+\gamma(n)}^-, \tilde{m}, U_{1+\gamma(n)}^+\}$ satisfies the two first properties of Definition 2.5. We are left with the third property.

Assume $\tilde{m} > 0$, we remark that $S^n_{U^-_{1+\gamma(n)}} - \max_{0 \leqslant t \leqslant m}\left(S^n_t\right) \leqslant \gamma(n) \Rightarrow \max_{0 \leqslant t \leqslant m}\left(S^n_t\right) \geqslant 1$ moreover $\max_{0 \leqslant t \leqslant \tilde{m}}\left(S^n_t\right) \leqslant 1+\gamma(n)$. Therefore

$$Q\left[S^n_{U^-_{1+\gamma(n)}} - \max_{0 \leqslant t \leqslant m}\left(S^n_t\right) \leqslant \gamma(n)\right] \leqslant Q\left[1 \leqslant \max_{0 \leqslant t \leqslant \tilde{m}}\left(S^n_t\right) \leqslant 1+\gamma(n)\right]. \tag{A.3}$$

Using (B.32) and Lemma (B.4), it is easy to prove that there exists $n_1 \equiv n_1(\gamma, \sigma, \mathbb{E}\left[|\epsilon_0|^3\right])$ such that for all $n > n_1$

$$Q\left[S^n_{\tilde{m}} \leqslant -\gamma(n)\right] \geqslant 1 - \frac{\log_2 n}{\log n}\left(\gamma + \mathcal{O}\left(\frac{1}{\log_2 n}\right)\right) . \tag{A.4}$$

Let us denote $\mathcal{A} = \{1 \leqslant \max_{0 \leqslant t \leqslant \tilde{m}}\left(S^n_t\right) \leqslant 1+\gamma(n),\ S^n_{\tilde{m}} \leqslant -\gamma(n)\}$, by A.3 and A.4 we have

$$Q\left[S^n_{U^-_{1+\gamma(n)}} - \max_{0 \leqslant t \leqslant m}\left(S^n_t\right) \leqslant \gamma(n)\right] \leqslant Q\left[\mathcal{A}\right] + \frac{\log_2 n}{\log n}\left(\gamma + \mathcal{O}\left(\frac{1}{\log_2 n}\right)\right) \tag{A.5}$$

Let us define

$$W_{\gamma(n)} = \begin{cases} \inf\{m \in \mathbb{N}^*,\ S^n_m \in [1, 1+\gamma(n)]\} , \\ +\infty, \text{ if such } m \text{ does not exist.} \end{cases} \tag{A.6}$$

Denote $\mathcal{A}' = \bigcup_{j > W_{\gamma(n)}}\left\{S^n_j \leqslant -\gamma(n),\ \bigcap_{k=W_{\gamma(n)}+1}^{j}\left\{S^n_k < 1+\gamma(n)\right\}\right\}$, we have $\mathcal{A} \subset \mathcal{A}'$ so $Q\left[\mathcal{A}\right] \leqslant Q\left[\mathcal{A}'\right]$. Making a partition on the values of $W_{\gamma(n)}$, using that $\{W_{\gamma(n)} = r\} \Rightarrow \{S^n_r \in [1, 1+\gamma(n)]\}$ and the strong Markov property we get

$$Q\left[\mathcal{A}'\right] \leqslant \sup_{1-\gamma(n) \leqslant x \leqslant 1}\left(Q\left[U^-_{\gamma(n)+x} < U^+_{1+\gamma(n)-x}\right]\right) \sum_{r=0}^{+\infty} \int_1^{1+\gamma(n)} Q\left[W_{\gamma(n)} = r,\ S^n_r \in dx\right] \tag{A.7}$$

$$\leqslant Q\left[U^-_1 < U^+_{2\gamma(n)}\right] . \tag{A.8}$$

Using Lemma B.4, we get that there exists $n_2 \equiv n_2(\sigma, \mathbb{E}[|\epsilon_0|^3])$ such that for all $n > n_2$

$$Q\left[U^-_1 < U^+_{2\gamma(n)}\right] \leqslant \frac{2\log_2 n}{\log n}\left(\gamma + O\left(\frac{1}{\log_2 n}\right)\right) . \tag{A.9}$$

Collecting what we did above and taking $n_0 = n_1 \vee n_2$ we get the lemma. ∎

Proof of Proposition 3.1. Let us prove 3.1, noticing that $\tilde{M}_0 \leqslant U^+_{1+\gamma(n)}$, and using remark (B.32), for all $G > 0$ we get

$$Q\left[\tilde{M}_0 > (\sigma^{-1}\log n)^2 \log_2 n\right] \leqslant Q\left[U^+_{1+\gamma(n)} \wedge U^-_G > (\sigma^{-1}\log n)^2\right] \\ +Q\left[U^+_1 \geqslant U^-_G\right] . \tag{A.10}$$

Taking $G = \left(\frac{2\log_2 n}{h_1^2 \log_3 n}\right)^{1/2}$ with $h_1 > 0$ and using (B.18), we get that there exists $n_1 \equiv n_1(h_1, \sigma, \mathbb{E}_Q\left[|\epsilon_0|^3\right])$ such that for all $n > n_1$

$$Q\left[U^+_{1+\gamma(n)} \wedge U^-_G > E(\log n)^2\right] \leqslant 2q_1^{\frac{h_1}{16}\log_3 n} , \tag{A.11}$$

where $q_1 < 0.7$. Choosing correctly the numerical constant h_1 we get for all $n > n_1$:

$$Q\left[U^+_{1+\gamma(n)} \wedge U^-_G > (\sigma^{-1}\log n)^2 \log_2 n\right] \leqslant \frac{1}{\log_2 n} . \tag{A.12}$$

Taking $D = \log n$ in (B.19) we get for all $n > n_1$

$$Q\left[U^+_{1+\gamma(n)} \geqslant U^-_G\right] \leqslant \frac{1}{G} + \mathcal{O}\left(\frac{(\log_2 n)^{3/2}}{\log n}\right). \tag{A.13}$$

Using (A.10), (A.12), (A.13) and the expression of G we get 3.1, the proof of 3.2 is similar. ∎

We recall that for all $\kappa \in]0, \kappa^+[$, $C \equiv C(\kappa) = \mathbb{E}_Q\left[e^{\kappa\epsilon_0}\right] \vee \mathbb{E}_Q\left[e^{-\kappa\epsilon_0}\right] < +\infty$.

Proof of Lemma 3.2. Denote

$$A_0 = \left\{\tilde{M}_0 \geqslant (\sigma^{-1}\log n)^2 \log_2 n, \tilde{M}'_0 \leqslant -(\sigma^{-1}\log n)^2 \log_2 n\right\}. \tag{A.14}$$

Let $u_n = \left[((\log n)^2)(65\sigma^2(\log_2 n))^{-1}\right] + 1$ and v_n a sequence such that $u_n \times v_n = [(\sigma^{-1}\log n)^2 \log_2 n] + 1$. Using 3.1 we know that there exists $n'_0 \equiv n'_0\left(\epsilon, \sigma, \mathbb{E}_Q\left[|\epsilon_0|^3\right]\right)$ such that for all $n > n'_0$

$$Q\left[\tilde{M}_0 - \tilde{m}_0 \leqslant u_n\right] \leqslant Q\left[\tilde{M}_0 - \tilde{m}_0 \leqslant u_n,\ A_0\right] + h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} . \tag{A.15}$$

We recall that, in all this work, h is a strictly positive numerical constant that can grow from line to line if needed. Let us denote $B_{n,\sigma} = \{-[(\sigma^{-1}\log n)^2\log_2 n]-1, [(\sigma^{-1}\log n)^2\log_2 n], \dots, [(\sigma^{-1}\log n)^2\log_2 n]+1\}$, by definition $S_{\tilde{M}_0} - S_{\tilde{m}_0} \geqslant \log n$, so

$$Q\left[\tilde{M}_0 - \tilde{m}_0 \leqslant u_n,\ A_0\right] \leqslant Q\left[\max_{m\in B_{n,\sigma}} \max_{m\leqslant l\leqslant m+u_n} \max_{m\leqslant j\leqslant m+u_n} \left(|S_l - S_j|\right) \geqslant \log n\right]. \quad (A.16)$$

Making similar computations to the ones did in the proof of B.4 we get that there exists $n_1 \equiv n_1(\sigma, C, \kappa)$ such that for all $n > n_1$,

$$Q\left[\max_{m\in B_{n,\sigma}} \max_{m\leqslant l\leqslant m+u_n} \max_{m\leqslant j\leqslant m+u_n} \left(|S_l - S_j|\right) \geqslant \log n\right] \leqslant \frac{4\log_2 n}{\sigma^2(\log n)^{1/33}}, \quad (A.17)$$

using (A.15), (A.16), (A.15) and taking $n_0 = n_0' \vee n_1$ we get 3.13. Similar computations give 3.14. ▮

The following result is essential to the proof of the other good properties.

Minimal distance between the two points of one refinement (property 3.25)

Lemma A.1. There exists $h > 0$ such that if (2.2), (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\sigma, \kappa, \mathbb{E}\left[|\epsilon_0|^3\right], C, \gamma)$ such that for all $n > n_0$

$$Q\left[\bigcup_{i=1}^{r'}\left\{\tilde{M}_i' - \tilde{m}_i' \leqslant b_n\right\}\right] \leqslant h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} + \mathcal{O}\left(\frac{\log_2 n}{(\log n)^{1/33}}\right), \quad (A.18)$$

$$Q\left[\bigcup_{i=1}^{r}\left\{\tilde{M}_i - \tilde{m}_i \leqslant b_n\right\}\right] \leqslant h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} + \mathcal{O}\left(\frac{\log_2 n}{(\log n)^{1/33}}\right). \quad (A.19)$$

b_n is given in (3.3), $\tilde{M}'_.$, $\tilde{m}'_.$ $\tilde{M}_.$ and $\tilde{m}_.$ have been defined Section 3.1.

Remark A.2. This lemma shows that the distance between two points obtained by the operation of refinement is larger than b_n.

Proof. Let $\kappa \in]0, \kappa^+[$ and $\gamma > 0$. Recalling (3.4) and (3.5), let us denote

$$A_1 = \bigcup_{i=1}^{r'} \left\{ \tilde{M}_i' - \tilde{m}_i' \leqslant b_n \right\}, \tag{A.20}$$

$$A_2 = \bigcup_{l=-[k_n]-1}^{[k_n]+1} \bigcup_{j=l+[l_n]}^{[k_n]+1} \left\{ \max_{(l+1)b_n \leqslant w<z \leqslant jb_n} (S_z - S_w) \leqslant \max_{lb_n \leqslant m \leqslant (j+1)b_n} \max_{m \leqslant u<v \leqslant m+b_n} (S_v - S_u) \right\}. \tag{A.21}$$

Denoting $C_1 = \bigcap_{j=0}^{r'} \bigcup_{l=-[k_n]-1}^{[k_n]+1} \left\{ \tilde{M}_j' \in [lb_n, (l+1)b_n] \right\}$ and $D_1 = \bigcup_{i=1}^{r'} \bigcup_{l=-[k_n]-1}^{[k_n]+1} \left\{ \tilde{M}_i' - \tilde{m}_i' \leqslant b_n,\ \tilde{M}_{i-1}' \in [lb_n, (l+1)b_n] \right\}$, it is clear that $\{A_1, C_1\} \subset \{D_1\}$. Now denoting $C_2 = \bigcap_{i=0}^{r'-1} \left\{ \tilde{M}_i' \leqslant \tilde{m}_0 - l_n b_n \right\}$ and $D_2 = \bigcup_{i=1}^{r'} \bigcup_{l=-[k_n]-1}^{[k_n]+1} \left\{ \tilde{M}_i' - \tilde{m}_i' \leqslant b_n,\ \tilde{M}_{i-1}' \in [lb_n, (l+1)b_n],\ \tilde{M}_{i-1}' \leqslant \tilde{m}_0 - l_n b_n \right\}$, we easily get that $\{D_1, C_2\} \subset D_2$. Finally, denoting $C_3 = \bigcup_{l=-[k_n]-1}^{[k_n]+1} \{\tilde{m}_0 \in [lb_n, (l+1)b_n]\}$, $D_3 = \bigcup_{i=1}^{r'} \bigcup_{l=-[k_n]-1}^{[k_n]+1} \bigcup_{j=l+[l_n]}^{[k_n]+1} \left\{ \tilde{M}_i' - \tilde{m}_i' \leqslant b_n,\ \tilde{M}_{i-1}' \in [lb_n, (l+1)b_n], \tilde{m}_0 \in [b_n j, b_n (j+1)] \right\}$ and noticing that $\left\{ \tilde{M}_{i-1}' \leqslant \tilde{m}_0 - l_n b_n, \tilde{M}_{i-1}' \in [lb_n, (l+1)b_n] \right\} \subset \{\tilde{m}_0 \geqslant lb_n + l_n b_n\}$, we get that $\{D_2, C_3\} \subset D_3$. Moreover, if we make a refinement of $\{\tilde{M}_{i-1}, \tilde{m}_0\}$, we get the points $\tilde{M}_i'$ and $\tilde{m}_i'$ such that $S_{\tilde{M}_i'} - S_{\tilde{m}_i'} = \max_{\tilde{M}_{i-1}' \leqslant w<z \leqslant \tilde{m}_0} (S_z - S_w)$, so $D_3 \subset A_2$. Therefore, we have:

$$Q[A_1] \leqslant Q[A_2] + Q[C_1^c] + Q[C_2^c] + Q[C_3^c]. \tag{A.22}$$

It is easy to see that $\{C_1^c \subset A_0^c\}$, $\{C_1^c \subset A_0^c\}$ and $C_2^c \subset \{\tilde{m}_0 - \tilde{M}_0' \geqslant (\log n)^2 (65\sigma^2 \log_2 n)^{-1}\}$ so using Proposition 3.23 and Lemma 3.2 we have some upper bounds for the three last probabilities of (A.22).
Now let us give an upper bound for $Q[A_2]$, first we introduce the following event, let $s > 0$

$$A_3 = \max_{-([k_n]+1)b_n \leqslant m \leqslant ([k_n]+1)b_n} \max_{m \leqslant l \leqslant m+b_n} \max_{m \leqslant j \leqslant m+b_n} \left(\left| S_l - S_j \right| \right) \leqslant g_n, \tag{A.23}$$

where $g_n = ((1+s)32\sigma^2 b_n \log k_n)^{1/2}$, we have

$$Q[A_2] \leqslant Q[A_2, A_3] + Q\left[A_3^c\right]. \tag{A.24}$$

Applying inequality (B.4), (taking $[L]+1=([k_n]+1)b_n$ and $\log K=\log(k_n)$) we get that there exists $n_1\equiv n_1(\sigma,s,\kappa,\mathbb{E}\left[|\epsilon_0|^3,C\right])$ such that for all $n>n_1$

$$Q\left[A_3^c\right]\leqslant\frac{4b_n}{k_n^{\frac{s}{2}}}. \tag{A.25}$$

We are left to estimate $Q[A_2,A_3]$, we have

$$Q[A_2,A_3]\leqslant\sum_{i=-[k_n]-1}^{[k_n]+1}Q\left[\bigcup_{j=i+[l_n]}^{[k_n]+1}\left\{\max_{(i+1)b_n\leqslant w<z\leqslant jb_n}(S_z-S_w)\leqslant g_n\right\}\right]. \tag{A.26}$$

We remark that the event $\left\{\max_{ib_n\leqslant w<z\leqslant jb_n}(S_z-S_w)\leqslant g_n\right\}$ is decreasing in j, so

$$Q[A_2,A_3]\leqslant\sum_{i=-[k_n]-1}^{[k_n]+1}Q\left[\max_{(i+1)b_n\leqslant w<z\leqslant(i+[l_n])b_n}(S_z-S_w)\leqslant g_n\right]. \tag{A.27}$$

Denoting $(a_n,n\in\mathbb{N}^*)$ and $(d_n,n\in\mathbb{N}^*)$ two strictly positive increasing sequence such that $[l_n]=d_n\times a_n$ we get by independence

$$Q[A_2,A_3]=2([k_n]+1)\left(Q\left[S_{a_nb_n}\leqslant g_n\right]\right)^{[d_n]-1}. \tag{A.28}$$

Now applying the Berry–Essen theorem to $Q\left[S_{a_nb_n}\leqslant g_n\right]$ and choosing $d_n=-2\frac{(\log(k_n+2))}{(\log(\int_1^{+\infty}e^{-x^2}/(2\pi)^{1/2}))}$, we obtain that there exists $n_2\equiv n_2(\sigma,\mathbb{E}_Q[|\epsilon_0|^3])$ such that for all $n>n_2$

$$Q[A_2,A_3]\leqslant\frac{2}{k_n}. \tag{A.29}$$

Finally, taking $s=4$ and using (A.24), (A.25) and (A.29) we get that there exists $n_3\equiv n_3(\sigma,\kappa,\mathbb{E}_Q\left[|\epsilon_0|^3\right],C,\gamma)\geqslant n_1\vee n_2$ such that for all $n>n_3$

$$Q[A_2]=\mathcal{O}\left(\frac{\log_2 n}{\log n}\right)^{1/2}. \tag{A.30}$$

Collecting (A.22) and (A.30) we get (A.18). Similar computations give (A.19). ∎

Corollary A.3. There exists $h>0$ such that if (2.2), (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds, for all $\gamma>0$ there exists $n_0 \equiv n_0(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right], C, \gamma)$ such that for all $n>n_0$

$$Q\left[r' \leqslant 2k_n+1\right] \geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{\log_2 n}{(\log n)^{1/33}}\right), \qquad \text{(A.31)}$$

$$Q\left[r \leqslant 2k_n+1\right] \geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{\log_2 n}{(\log n)^{1/33}}\right). \qquad \text{(A.32)}$$

r and r' have been defined Section 3.1 and k_n is given in (3.4).

Proof. This corollary is an easy consequence of Lemma A.1, the proof is omitted. ▮

Minimal distance between two maximums (properties (3.27) and (3.30))

Proposition A.4. There exists $h>0$ such that if (2.2), (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds, there exists $n_0 \equiv n_0(\sigma, \kappa, \mathbb{E}\left[|\epsilon_0|^3\right], \mathbb{E}\left[\epsilon_0^4\right], C, \gamma)$ such that for all $n>n_0$

$$Q\left[\bigcap_{i=0}^{r-1}\left\{\eta_{i,i+1} \geqslant \gamma(n)\right\}\right] \geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \qquad \text{(A.33)}$$

$$Q\left[\bigcap_{i=0}^{r'-1}\left\{\eta'_{i,i+1} \geqslant \gamma(n)\right\}\right] \geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \qquad \text{(A.34)}$$

where $\gamma(n)$ is given a the end of Definition 3.4, $\eta_{.,.}$ and $\eta'_{.,.}$ are given in (3.6).

Proof. Let us prove A.34. To prove this proposition we will use the Lemma A.1. Let $n>3$, and $\gamma>0$, we recall the following notations $b_n = \left[(\gamma)^{1/2}(\log n \log_2 n)^{3/2}\right]+1$, $k_n = ((\sigma^{-1}\log n)^2 \log_2 n)/b_n$. Let us denote

$$A = \bigcap_{i=0}^{r'}\left\{-(\sigma^{-1}\log n)^2\log_2 n \leqslant \tilde{M}'_i \leqslant (\sigma^{-1}\log n)^2 \log_2 n\right\}, \qquad \text{(A.35)}$$

$$A_1 = \bigcup_{i=1}^{r'} \bigcup_{j=-[k_n]-1}^{[k_n]+1} \left\{m'_i \in [b_n j, b_n(j+1)],\ \ M'_i \in [b_n j, b_n(j+1)]\right\}, \qquad \text{(A.36)}$$

$$A_2 = \bigcup_{i=1}^{r'} \bigcup_{j=-[k_n]-1}^{[k_n]+1} \left\{ M'_i \in [b_n j, b_n(j+1)],\ \ M'_{i+1} \in [b_n j, b_n(j+1)] \right\}, \tag{A.37}$$

$$A_3 = \bigcup_{i=0}^{r'-1} \left\{ 0 \leqslant \eta'_{i,i+1} \leqslant \gamma(n) \right\}. \tag{A.38}$$

We have $Q[A_3] \leqslant Q[A_3, A_1^c, A] + Q[A_1] + Q[A^c]$, moreover $A \subset A_0$ (see A.14) and $A_1 \subset \bigcup_{i=1}^{r'} \left\{ \tilde{M}'_i - \tilde{m}'_i \leqslant b_n \right\}$, therefore using Lemma (3.23) and the inequality (A.18) we get that there exists $h > 0$ and n_1 such that for all $n > n_1$, $Q[A_3] \leqslant Q[A_3, A_1^c, A] + h((\log_2 n)/(\log n))^{1/2}$. Let us denote $L_{i,j}(n) = \max_{b_n i \leqslant k \leqslant b_n(i+1)} \left(S_k^n\right) - \max_{b_n j \leqslant l \leqslant b_n(j+1)} \left(S_l^n\right)$, define

$$A_4 = \bigcup_{i=-[k_n]-1}^{[k_n]+1} \bigcup_{j=i+1}^{[k_n]+1} \left\{ 0 \leqslant L_{i,j}(n) \leqslant \gamma(n) \right\}, \tag{A.39}$$

by definition of the refinements we have $\tilde{M}'_i < \tilde{M}'_{i+1}$ and $S_{\tilde{M}'_i} > S_{\tilde{M}'_{i+1}}$, $\forall i\ 0 \leqslant i \leqslant r' - 1$, therefore $\{A_3,\ A_2^c,\ A\} \subset A_4$ then $Q\left[A_3,\ A_2^c,\ A\right] \leqslant Q\left[A_4\right]$. Finally, we get that for all $n > n_3$

$$Q[A_3] \leqslant Q[A_4] + h((\log_2 n)/(\log n))^{1/2} \tag{A.40}$$

Denoting

$$A_5 = \bigcup_{i=-[k_n]-1}^{[k_n]+1} \bigcup_{j=i+2}^{[k_n]+1} \left\{ 0 \leqslant L_{i,j}(n) \leqslant \gamma(n) \right\}, \tag{A.41}$$

$$A_6 = \bigcup_{i=-[k_n]-1}^{[k_n]+1} \left\{ 0 \leqslant L_{i,i+1}(n) \leqslant \gamma(n) \right\}. \tag{A.42}$$

we have that

$$Q\left[A_4\right] = Q\left[A_5\right] + Q\left[A_6\right]. \tag{A.43}$$

Now we estimate the two probability $Q\left[A_5\right]$ and $Q\left[A_6\right]$ in (respectively) Lemma A.5 and A.6. For the proof of these lemmata we have used the paper in preparation of Cassandro *et al.*[44]

Lemma A.5. Assume (2.2), (2.3) and (2.4), for all $\gamma > 0$ there exists $n_0' \equiv n_0'\left(\sigma, \gamma, \mathbb{E}\left[\epsilon_0^4\right]\right)$ such that for all $n > n_0'$

$$Q\left[A_5\right] \leqslant 10\left(\frac{\pi}{\sigma^2}\right)^{1/2} \frac{\gamma \log_2 n}{(b_n)^{1/2}} \left([k_n]+1\right)^{3/2} \tag{A.44}$$

where k_n is given by (3.4), b_n by (3.3).

Proof. We have

$$Q\left[A_5\right] \leqslant \sum_{i=-[k_n]-1}^{[k_n]+1} \sum_{j=i+2}^{[k_n]+1} Q\left[\left\{0 \leqslant L_{i,j}(n) \leqslant \gamma(n)\right\}\right]. \tag{A.45}$$

Now we give an upper bound for $\sum_{i=0}^{[k_n]+1} \sum_{j=i+2}^{[k_n]+1} Q\left[\left\{0 \leqslant L_{i,j}(n) \leqslant \gamma(n)\right\}\right]$. Denoting $Z_{i+1,j}(n) = -\sum_{l=b_n(i+1)+1}^{b_n j} \epsilon_l$ and $Y = -\min_{i b_n \leqslant k \leqslant (i+1)b_n} \sum_{m=k}^{(i+1)b_n} \epsilon_m - \max_{j b_n+1 \leqslant k \leqslant (j+1)b_n} \sum_{m=j b_n+1}^{k} \epsilon_m$, it is easy to see that for all $i \geqslant 0$, $L_{i,j}(n) = (Z_{i+1,j}(n) + Y)/(\log n)$. Therefore we have

$$Q\left[0 \leqslant L_{i,j}(n) \leqslant \gamma(n)\right] = \int_{\mathbb{R}} Q\left[0 \leqslant Z_{i+1,j}(n) - y \leqslant \gamma(n)\log n,\ \ Y \in dy\right]. \tag{A.46}$$

$Z_{i+1,j}(n)$ and Y are independent so

$$\begin{aligned}\int_{\mathbb{R}} & Q\left[0 \leqslant Z_{i+1,j}(n) - y \leqslant \gamma(n)\log n,\ \ Y \in dy\right] \\ &\leqslant \sup_y \left(Q\left[y \leqslant Z_{i+1,j}(n) \leqslant \gamma(n)\log n + y\right]\right).\end{aligned} \tag{A.47}$$

To estimate this last term we use the following concentration inequality (see Lecam, pp. 401–413)[45]

$$\sup_y \left(Q\left[y \leqslant Z_{i+1,j}(n) \leqslant \gamma(n)\log n + y\right]\right) \leqslant \frac{2(\pi)^{1/2}}{Z}, \tag{A.48}$$

where $Z^2 \equiv Z^2(\gamma(n)) = \sum_{l=1}^{b_n(j-i-1)} \mathbb{E}\left[1 \wedge H_s^2\right]$, $H_s = \frac{\epsilon_l^s}{\gamma(n)\log n}$ and $\epsilon_l^s = \epsilon_l - \epsilon_s'$, ϵ_l' is independent and identically distributed to ϵ_l. We have $\mathbb{E}\left[1 \wedge (H_s)^2\right] \geqslant$

$(\gamma(n)\log n)^{-2}\mathbb{E}\left[\left(\epsilon_l^s\right)^2 \mathbb{I}_{1>H_s}\right]$. Noticing that $\mathbb{E}\left[\left(\epsilon_l^s\right)^2 \mathbb{I}_{1>H_s}\right] = \mathbb{E}\left[\left(\epsilon_l^s\right)^2\right] - \mathbb{E}\left[\left(\epsilon_l^s\right)^2 \mathbb{I}_{1\leqslant H_s}\right]$ we get by Schwarz inequality and Markov inequality

$$\mathbb{E}\left[\left(\epsilon_l^s\right)^2 \mathbb{I}_{1>H_s}\right] \geqslant 2\sigma^2 - \left(\mathbb{E}\left[\left(\epsilon_l^s\right)^4\right]^{1/2} (2\sigma^2)^{1/2}\right)(\gamma \log_2 n)^{-1}. \quad (A.49)$$

We deduce that there exists $n_0' \equiv n_0'\left(\sigma, \gamma, \mathbb{E}\left[\epsilon_0^4\right]\right)$ such that for all $n > n_0'$, $\mathbb{E}\left[1 \wedge (H_s)^2\right] \geqslant 3\sigma^2/(2(\gamma(n)\log n)^2)$, therefore for all $n > n_0'$

$$Z \geqslant \sqrt{\frac{3}{2}\sigma^2} \frac{\sqrt{b_n(j-i-1)}}{\gamma(n)\log n}. \quad (A.50)$$

Inserting (A.50) in (A.48) and using (A.47) and (A.46) we obtain for all $n > n_0'$

$$Q\left[0 \leqslant L_{i,j}(n) \leqslant \gamma(n)\right] \leqslant \left(\frac{8\pi}{3\sigma^2}\right)^{1/2} \frac{\gamma(n)\log n}{(b_n)^{1/2}(j-i-1)^{1/2}}. \quad (A.51)$$

Therefore, using (A.51) for all $n > n_0'$ we have

$$\sum_{i=0}^{[k_n]+1} \sum_{j=i+2}^{[k_n]+1} Q\left[\left\{0 \leqslant L_{i,j}(n) \leqslant \gamma(n)\right\}\right] \leqslant \frac{5}{2}\left(\frac{\pi}{\sigma^2}\right)^{1/2} \frac{\gamma \log_2 n}{(b_n)^{1/2}} ([k_n]+1)^{3/2}. \quad (A.52)$$

Making similar computations for the case $i < 0$ we get a similar result, so we get Lemma A.5. ∎

Constraint on k_n and b_n. Now we can justify the choice for b_n and k_n, recalling that $k_n \times b_n = (\sigma^{-1}\log n)^2 \log_2 n$ we want that

$$\left(\frac{\pi}{\sigma^2}\right)^{1/2} \frac{\gamma \log_2 n}{(b_n)^{1/2}} ([k_n]+1)^{3/2}, \quad (A.53)$$

be close to 0 but b_n small. Using that $b_n = \left[(\gamma)^{1/2}(\log n \log_2 n)^{3/2}\right] + 1$, we get that there exists $h_1 \equiv h_1(\sigma, \gamma) > 0$ and n_2 such that for all $n > n_2$,

$$10\left(\frac{\pi}{\sigma^2}\right)^{1/2} \frac{\gamma \log_2 n}{(b_n)^{1/2}} ([k_n]+1)^{3/2} \leqslant h_1 \left(\frac{1}{\log_2 n}\right)^{1/2}. \quad (A.54)$$

So using (A.54) and Lemma A.5, we get that there exists $n_1' \equiv n_1'(\sigma, \gamma, \mathbb{E}[\epsilon_0^4]) \geqslant n_0' \vee n_2$ such that for all $n > n_1'$

$$Q\left[\bigcup_{i=-[k_n]-1}^{[k_n]+1} \bigcup_{j=i+2}^{[k_n]+1} \left\{\max_{b_n i \leqslant k \leqslant b_n(i+1)} \left(S_k^n\right) - \max_{b_n j \leqslant l \leqslant b_n(j+1)} \left(S_l^n\right) \leqslant \gamma(n)\right\}\right] \leqslant h_1 \left(\frac{1}{\log_2 n}\right)^{1/2}. \tag{A.55}$$

Now we prove the following lemma.

Lemma A.6. Assume (2.2), (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 0$ there exists $n_0'' \equiv n_0''(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right], \mathbb{E}\left[\epsilon_0^4\right], C, \gamma)$ such that for all $n > n_0''$

$$Q[A_6] \leqslant \frac{(2[k_n]+3)(\log_2 n)^{5/2}}{(b_n)^{1/2}} \left(2\gamma + \left(\frac{16\pi}{3\sigma^2}\right) \frac{\gamma}{\sigma(\log_2 n)^{3/2}}\right). \tag{A.56}$$

Proof. We have

$$Q\left[\bigcup_{i=-[k_n]-1}^{[k_n]+1} \left\{0 \leqslant L_{i,i+1}(n) \leqslant \gamma(n)\right\}\right] \leqslant \sum_{i=-[k_n]-1}^{[k_n]+1} Q\left[0 \leqslant L_{i,i+1}(n) \leqslant \gamma(n)\right]. \tag{A.57}$$

Using the fact that we can write $\max_{b_n(i+1) \leqslant l \leqslant b_n(i+2)} \left(S_l^n\right) = X + \max_{b_n(i+1)+1 \leqslant l \leqslant b_n(i+2)} \left(\sum_{l=b_n(i+1)}^{l}\right)$ with $X \in \sigma\left(\epsilon_1, \dots, \epsilon_{b_n(i+1)}\right)$ and $Y \equiv \max_{b_n i \leqslant k \leqslant b_n(i+1)} \left(S_k^n\right) \in \sigma\left(\epsilon_1, \dots, \epsilon_{b_n(i+1)}\right)$ we easily get by independence that

$$Q\left[0 \leqslant L_{i,i+1}(n) \leqslant \gamma(n)\right] \leqslant \sup_x \left(Q\left[x \leqslant \max_{1 \leqslant k \leqslant b_n} \left(S_k^n\right) \leqslant x + \gamma(n)\right]\right), \tag{A.58}$$

replacing this in (A.57), we get

$$Q\left[\bigcup_{i=-[k_n]-1}^{[k_n]+1} \left\{0 \leqslant L_{i,i+1}(n) \leqslant \gamma(n)\right\}\right] \leqslant (2[k_n]+3) \sup_x \left(Q\left[x \leqslant \max_{1 \leqslant k \leqslant b_n} \left(S_k^n\right) \leqslant x + \gamma(n)\right]\right). \tag{A.59}$$

To estimate $\sup_x \left(Q\left[x \leqslant \max_{1\leqslant k\leqslant b_n}\left(S_k^n\right) \leqslant x+\gamma(n)\right]\right)$ we remark that

$$Q\left[x \leqslant \max_{1\leqslant k\leqslant b_n}\left(S_k^n\right) \leqslant x+\gamma(n)\right] = Q\left[U_x^+ \leqslant b_n \leqslant U_{x+\gamma(n)}^+\right] \tag{A.60}$$

$$= Q\left[U_x^+ \leqslant \frac{b_n}{2},\ U_{x+\gamma(n)}^+ \geqslant b_n\right] \tag{A.61}$$

$$+Q\left[\frac{b_n}{2} < U_x^+ \leqslant b_n \leqslant U_{x+\gamma(n)}^+\right]. \tag{A.62}$$

We have to estimate the two probability in (A.61) and (A.62). We begin with (A.62), we remark that

$$\frac{b_n}{2} < U_x^+ \leqslant b_n \leqslant U_{x+\gamma(n)}^+ \Rightarrow x \leqslant \max_{b_n/2\leqslant k\leqslant b_n}\left(S_k^n\right) \leqslant x+\gamma(n), \tag{A.63}$$

from this we deduce by the concentration inequality (see equations (A.48)–A.51) that there exists $n_3 \equiv n_3(\sigma, \mathbb{E}\left[\epsilon_0^4\right])$ such that for all $n > n_3$

$$Q\left[\frac{b_n}{2} < U_x^+ \leqslant b_n \leqslant U_{x+\gamma(n)}^+\right] \leqslant \sup_y\left(Q\left[y \leqslant S_{b_n/2}^n \leqslant y+\gamma(n)\right]\right)$$
$$\leqslant \left(\frac{16\pi}{3\sigma^2}\right)^{1/2} \frac{\gamma \log_2 n}{(b_n)^{1/2}}. \tag{A.64}$$

Now we estimate the probability in (A.61), by the strong Markov property we have

$$Q\left[U_x^+ \leqslant \frac{b_n}{2},\ U_{x+\gamma(n)}^+ \geqslant b_n\right]$$
$$=\sum_{l=0}^{b_n/2}\int_x^{x+\gamma(n)} Q\left[U_x^+ = l, S_l \in dy\right] Q\left[\ U_{x+\gamma(n)-y}^+ \geqslant b_n - l\right], \tag{A.65}$$

moreover $x-y \leqslant 0$, therefore $Q\left[\ U_{x+\gamma(n)-y}^+ \geqslant b_n-l\right] \leqslant Q\left[\ U_{\gamma(n)}^+ \geqslant b_n-l\right]$, so we get

$$Q\left[U_x^+ \leqslant \frac{b_n}{2},\ U_{x+\gamma(n)}^+ \geqslant b_n\right] \leqslant Q\left[\ U_{\gamma(n)}^+ \geqslant b_n/2\right]. \tag{A.66}$$

To estimate this probability we use Remark B.32 and Lemma B.4 (taking $c=\frac{\gamma\log_2 n}{\log n}$, $a=\frac{(b_n)^{1/2}}{\log n(\log_2 n)^{3/2}}$, $L=b_n/2$ and $D=\log n$), we get that there exists n_4 such that for all $n>n_4$

$$Q\Big[\ U^+_{\gamma(n)}\geqslant b_n/2\Big]\leqslant\frac{2\gamma(\log_2 n)^{5/2}}{(b_n)^{1/2}}. \tag{A.67}$$

Inserting (A.64) and (A.67) in (respectively) (A.61) and (A.62) and using (A.59) we get for all $n>n_4$

$$\begin{aligned}Q&\left[\bigcup_{i=-[k_n]-1}^{[k_n]+1}\{0\leqslant L_{i,i+1}(n)\leqslant\gamma(n)\}\right]\\&\leqslant\frac{(2[k_n]+3)(\log_2 n)^{5/2}}{(b_n)^{1/2}}\left(2\gamma+\left(\frac{16\pi}{3\sigma^2}\right)\frac{\gamma}{\sigma(\log_2 n)^{3/2}}\right),\end{aligned} \tag{A.68}$$

taking $n''_0=n_3\vee n_4$ we get Lemma A.6 . ▮

Recalling (3.3) and (3.4) we get from Lemma A.6, that for all $\kappa\in\,]0,\kappa^+[$, $\gamma>0$ there exists $n''_1\equiv n''_1(\sigma,\kappa,\mathbb{E}\left[|\epsilon_0|^3\right],\mathbb{E}\left[\epsilon_0^4\right],C,\gamma)\geqslant n''_0$ such that for all $n>n''_1$

$$Q\left[\bigcup_{i=-[k_n]-1}^{[k_n]+1}\{0\leqslant L_{i,i+1}(n)\leqslant\gamma(n)\}\right]=\mathcal{O}\left(\frac{(\log_2 n)^{1+3/4}}{(\log n)^{1/4}}\right). \tag{A.69}$$

To end the proof of Proposition A.4, we collect (A.69), (A.55), (A.43), and finally (A.40), and we take $n_0=n_1\vee n'_1\vee n''_1$. We get (A.33) with similar computations. ▮

Minimal distance between the maximum and the minimum of one refinement (properties (3.28) and (3.31))

Proposition A.7. There exists $h>0$ such that if (2.2), (2.3) hold and for all $\kappa\in]0,\kappa^+[$ (2.4) holds, for all $\gamma>0$ there exists $n_0\equiv n_0(\sigma,\mathbb{E}\left[|\epsilon_0|^3\right],\mathbb{E}\left[\epsilon_0^4\right],C,\gamma)$ such that for all $n>n_0$

$$Q\left[\bigcap_{i=0}^{r-1}\{\delta_{i+1,i+1}\geqslant\gamma(n)\}\right]\geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}-\mathcal{O}\left(\frac{1}{\log_2 n}\right), \tag{A.70}$$

$$Q\left[\bigcap_{i=0}^{r'-1}\{\delta'_{i+1,i+1}\geqslant\gamma(n)\}\right]\geqslant 1-h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}-\mathcal{O}\left(\frac{1}{\log_2 n}\right), \tag{A.71}$$

where $\gamma(n)$ is given at the end of Definition 3.4, $\delta_{\cdot,\cdot}$ and $\delta'_{\cdot,\cdot}$ are given in (3.6).

Proof. First, we remark that by construction the event $\{\delta_{i+1,i+1} \geqslant \gamma(n)\}$ decrease in i, so $Q\left[\bigcap_{i=0}^{r-1}\{\delta_{i+1,i+1} \geqslant \gamma(n)\}\right] = Q\left[\delta_{r,r} \geqslant \gamma(n)\right]$, then we use the same method used to prove Proposition A.4. ∎

Minimal distance between a minimum and $S_{\tilde{m}_0}$ (properties (3.29) and (3.29))

Proposition A.8. There exists $h > 0$ such that if (2.2), (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right], \mathbb{E}\left[\epsilon_0^4\right], C, \gamma)$ such that for all $n > n_0$

$$Q\left[\bigcap_{i=0}^{r-1}\{\mu_{i+1,0} \geqslant \gamma(n)\}\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \quad (A.72)$$

$$Q\left[\bigcap_{i=0}^{r'-1}\{\mu'_{i+1,0} \geqslant \gamma(n)\}\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \quad (A.73)$$

where $\gamma(n)$ is given at the end of Definition 3.4, $\mu_{\cdot,\cdot}$ and $\mu'_{\cdot,\cdot}$ are given in (3.6).

The proof of this proposition is similar to the proof of Proposition A.4 and is omitted.

Control of the first and the last refinement (properties (3.33), (3.34), (3.36) and (3.35))

Proposition A.9. There exists $h > 0$ such that if (2.2), (2.3) hold and for all $\kappa \in]0, \kappa^+[$ (2.4) holds, for all $\gamma > 0$ there exists $n_0 \equiv n_0(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right], \mathbb{E}\left[\epsilon_0^4\right], C, \gamma)$ such that for all $n > n_0$

$$Q\left[\delta_{1,1} \leqslant 1 - \gamma(n)\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \quad (A.74)$$

$$Q\left[\delta'_{1,1} \leqslant 1 - \gamma(n)\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{1}{\log_2 n}\right), \quad (A.75)$$

$$Q\left[\delta_{r,r} \leqslant (\log(q_n))(\log n)^{-1}\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{(\log_2 n)^{11/2}}{(\log n)^{1/66}}\right), \tag{A.76}$$

$$Q\left[\delta'_{r',r'} \leqslant (\log(q_n))(\log n)^{-1}\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2} - \mathcal{O}\left(\frac{(\log_2 n)^{11/2}}{(\log n)^{1/66}}\right), \tag{A.77}$$

where $\gamma(n)$ and q_n are given at the end of Definition 3.4.

Proof. *Let us prove A.74*, by construction $\delta_{1,1} \leqslant 1+\gamma(n)$. So we have to prove that the event $-\gamma(n) \leqslant \delta_{1,1}-1 \leqslant \gamma(n)$ has a probability very near 0, to do this make we make use similar computations used to prove Proposition A.4. A similar remark work for (A.75).

Let us prove A.76, by construction we have

$$\tilde{M}'_0 \leqslant \tilde{M}_r \leqslant \tilde{M}_0, \tag{A.78}$$

$$\tilde{M}_r - \tilde{m}_0 \leqslant l_n \times b_n. \tag{A.79}$$

Using (A.78) and Proposition 3.1, we know that there exists $n_1 \equiv n_1\left(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right]\right)$ such that for all $n > n_1$

$$Q\left[-(\sigma^{-1}\log n)^2 \log_2 n \leqslant \tilde{M}_r \leqslant (\sigma^{-1}\log n)^2 \log_2 n\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}. \tag{A.80}$$

Let us make the following chopping $\left[(\sigma^{-1}\log n)^2 \log_2 n + 1\right] = b'_n \times k'_n$ with $b'_n = [l_n \times b_n] + 1$, we have $\delta_{r,0} \geqslant \delta_{r,r}$, therefore, denoting $L'(n) = \max_{-b'_n \times k'_n \leqslant m \leqslant b'_n \times k'_n} \max_{m \leqslant j \leqslant m+b'_n} \max_{m \leqslant l \leqslant m+b'_n} \left(\left|S_l^n - S_j^n\right|\right)$

$$\left\{\begin{array}{l} -(\sigma^{-1}\log n)^2 \log_2 n \leqslant \tilde{M}_r \leqslant (\sigma^{-1}\log n)^2 \log_2 n \\ \text{and } \tilde{m}_0 - \tilde{M}_r \leqslant l_n \times b_n. \end{array}\right\} \Rightarrow \delta_{r,r} \leqslant \delta_{r,0} \leqslant L'(n). \tag{A.81}$$

From this and (A.80) we deduce that for all $n > n_1$ we have

$$Q\left[\delta_{r,r} \leqslant L'(n)\right] \geqslant 1 - h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2}. \tag{A.82}$$

Using (B.4) (with $K=k_n'$, $[L]+1=[(\sigma^{-1}\log n)^2\log_2 n]+1$, $B=b_n'$ and $s=4$) one can check that that there exists $n_2\equiv n_2\left(\sigma,s,\kappa,\mathbb{E}\left[|\epsilon_0|^3\right],C\right)$ such that for all $n>n_2$

$$Q\left[L'(n)>((1+s)32\sigma^2 b_n'\log k_n')^{1/2}\right]=\mathcal{O}\left(\frac{(\log_2 n)^{11/2}}{(\log n)^{1/66}}\right). \quad \text{(A.83)}$$

Using (A.82) and (A.83) we get that for all $n>n_2$

$$Q\left[\delta_{r,r}(\log n)\leqslant(160\sigma^2 b_n'\log k_n')^{1/2}\right]\geqslant 1-h\left((\log_3 n)(\log_2 n)^{-1}\right)^{1/2} -\mathcal{O}\left(\frac{(\log_2 n)^{11/2}}{(\log n)^{1/66}}\right). \quad \text{(A.84)}$$

Moreover, we remark that there exists $n_3\equiv n_3\,(\sigma,s,\kappa)$ such that for all $n>n_3$

$$160\sigma^2 b_n'\log k_n'\leqslant(200\sigma)^2(\gamma)^{1/2}(\log_2 n)^{7/2}(\log n)^{3/2}. \quad \text{(A.85)}$$

We get (A.76), taking $n_0=n_1\vee n_2\vee n_3$. Similar computations give the result for $\delta'_{r',r'}$. ∎

Proof for the property (3.24)

Lemma A.10. There exists $h>0$ such that if (2.2), (2.3) hold and for all $\kappa\in]0,\kappa^+[$ (2.4) holds, for all $\gamma>0$ there exists $n_0\equiv n_0(\gamma,\sigma,\mathbb{E}\left[|\epsilon_0|^3\right])$ such that for all $n>n_0$

$$Q\left[\tilde M_>\geqslant\tilde m_0+L_n\right]\leqslant h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}, \quad \text{(A.86)}$$

$$Q\left[\tilde M_<\leqslant\tilde m_0-L_n\right]\leqslant h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}, \quad \text{(A.87)}$$

see (3.15) for the definitions of $\tilde M_<$ and $\tilde M_>$ and Definition 3.4 for L_n one.

Proof. Denote $f(n)=(\log(q_n(\log n)^\gamma))/(\log n)$, where q_n is given at the end od Definition 3.4, we have

$$Q\left[\tilde M_>\geqslant\tilde m_0+L_n\right]\equiv Q\left[\inf\left\{m>\tilde m_0,\ S_m^n-S_{\tilde m_0}^n\geqslant f(n)\right\}\geqslant\tilde m_0+L_n\right], \quad \text{(A.88)}$$

$$= Q\left[\inf\left\{m > \tilde{m}_0,\ |S_m^n - S_{\tilde{m}_0}^n| \geqslant f(n)\right\} \geqslant \tilde{m}_0 + L_n\right], \tag{A.89}$$

because $\tilde{m}_0$ is a minimizer of the valley $\{\tilde{M}_0', \tilde{m}_0, \tilde{M}_0\}$ and by definition $\tilde{M}_0 \geqslant M_>$. Using Proposition 3.1, we know that there exists $n_1 \equiv n_1\left(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right]\right)$ such that for all $n > n_1$

$$Q\left[-(\sigma^{-1}\log n)^2\log_2 n \leqslant \tilde{m}_0 \leqslant (\sigma^{-1}\log n)^2\log_2 n\right] \geqslant 1 - h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}, \tag{A.90}$$

so for all $n > n_1$

$$Q\left[\inf\left\{m > \tilde{m}_0,\ |S_m^n - S_{\tilde{m}_0}^n| \geqslant f(n)\right\} \geqslant \tilde{m}_0 + L_n\right] \tag{A.91}$$

$$\leqslant \sum_{k=-[(\sigma^{-1}\log n)^2\log_2 n]-1}^{[(\sigma^{-1}\log n)^2\log_2 n]+1} Q\left[\inf\left\{m > k,\ |S_m^n - S_k^n| \geqslant f(n)\right\} \geqslant k + L_n\right] + h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}. \tag{A.92}$$

We get that for all $n > n_1$

$$Q\left[\inf\left\{m > \tilde{m}_0,\ |S_m^n - S_{\tilde{m}_0}^n| \geqslant f(n)\right\} \geqslant \tilde{m}_0 + L_n\right] \leqslant 2([(\sigma^{-1}\log n)^2\log_2 n] + 1)\, Q\left[U_{f(n)}^- \wedge U_{f(n)}^+ \geqslant L_n\right] + h\left(\frac{\log_3 n}{\log_2 n}\right)^{1/2}. \tag{A.93}$$

Applying inequality (B.18) we get that there exists $n_2 \equiv n_2\left(\sigma, \mathbb{E}\left[|\epsilon_0|^3\right]\right)$ such that for all $n > n_2$

$$Q\left[U_{f(n)}^- \wedge U_{f(n)}^+ \geqslant L_n\right] = \mathcal{O}\left(\frac{1}{\log n}\right). \tag{A.94}$$

Replacing this in (A.93) and using (A.89), we get (A.86) taking $n_0 = n_1 \vee n_2$. The proof of (A.87) is similar. ∎

Proof of Proposition 3.6

We only have to collect the results of the Lemmata 2.6, B.3 and A.10, of the Propositions 3.1, A.4, A.7, A.8 and A.9 and of the Corollary A.3.

APPENDIX B. STANDARD RESULTS ON SUMS OF I.I.D. RANDOM VARIABLES

We recall that for all $\kappa \in]0,\kappa^+[$, $C \equiv C(\kappa) = \mathbb{E}_Q\left[e^{\kappa\epsilon_0}\right] \vee \mathbb{E}_Q\left[e^{-\kappa\epsilon_0}\right] < +\infty$. In this section we recall some elementary results on sums of i.i.d. random variables satisfying the three hypothesis (2.2), (2.3) and (2.4). We will always work on the right of the origin, that means with $(S_m, m\in\mathbb{N})$, by symmetry we obtain the same results for $m\in\mathbb{Z}_-$.

The following lemma is an immediate consequence of Bernstein inequality (see Renyi[46]).

Lemma B.1. Assume (2.2), (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds. For all $q>0$ and $p>0$ such that $q<(\sigma^2 p)\wedge\left(\sigma^4 p/(2C)\right)$ we have

$$Q\left[|S_p|>q\right] \leqslant 2\exp\left\{-\frac{q^2}{2\sigma^2 p}\left(1-\frac{2qC}{\sigma^4 p}\right)\right\}, \tag{B.1}$$

For all $p>1$, $s>0$ and $k>1$ such that $\log k<(1+s)32\sigma^2 p$, for all $0\leqslant j\leqslant p$ we have

$$\begin{aligned} Q&\left[\left|S_p-S_j\right|>\left(32(1+s)\sigma^2 p\log k\right)^{1/2}\right] \\ &\leqslant 2\exp\left\{-\log k+\frac{(p-j)\log k}{(1+s)64p}+\frac{(p-j)(\log k)^{3/2}C}{((1+s)32\sigma^2 p)^{3/2}}\right\}. \end{aligned} \tag{B.2}$$

The following lemma gives an upper bound to the largest fluctuation of the potential $(S_r, r\in\mathbb{R})$ in a block of length B of a given interval.

Lemma B.2. Assume (2.2), (2.3) hold and for all $\kappa \in]0,\kappa^+[$ (2.4) holds. For all $s>0$, all integers $K>1$ and $B>1$ such that $\log K<\sigma^2\kappa^2 B$ we have

$$\begin{aligned} &Q\left[\max_{-K-1\leqslant i\leqslant K}\ \max_{iB\leqslant j\leqslant (i+1)B}\ \max_{iB\leqslant l\leqslant (i+1)B}\left(\left|S_l-S_j\right|\right)>((1+s)32\sigma^2 B\log K)^{1/2}\right] \\ &\quad\leqslant 2K^{-(s-\mathcal{O}((\log K)/B)^{1/2})}\left(1+\mathcal{O}\left(H_{K,B}\right)\right). \end{aligned} \tag{B.3}$$

where $H_{K,B} = K^{-(1-1/64-\mathcal{O}((\log K)/B)^{1/2})}$. For all $L > 1$, $K > 1$, all integers $B > 1$ such that $[L] + 1 = K \times B$ and all $s > 0$ such that $\log K < (1 + s)32\sigma^2\sigma^2\kappa^2 B$, we have

$$Q\left[\max_{-[L]-1 \leqslant m \leqslant [L]+1} \max_{m \leqslant l \leqslant m+B} \max_{m \leqslant j \leqslant m+B} \left(|S_l - S_j|\right) > ((1+s)32\sigma^2 B \log K)^{1/2}\right] \\ \leqslant 2(B+1)K^{-(s-\mathcal{O}((\log K)/B)^{1/2})}\left(1+\mathcal{O}\left(H_{K,B}\right)\right). \tag{B.4}$$

Proof. Let us prove (B.3), let $s > 0$, $K > 1$ and $B > 1$ two positive integers, denoting $q = ((1+s)32\sigma^2 B \log K)^{1/2}$. Using the fact that $(\alpha_i, i \in \mathbb{Z})$ are i.i.d. we get

$$Q\left[\max_{-K-1 \leqslant i \leqslant K} \max_{-iB \leqslant j \leqslant (i+1)B} \max_{iB \leqslant l \leqslant (i+1)B} \left(|S_l - S_j|\right) > q\right] \\ \leqslant 1 - \left(1 - Q\left[2 \max_{1 \leqslant j \leqslant B} \left(|S_j|\right) > q\right]\right)^{2K+2}. \tag{B.5}$$

By Ottaviani inequality (see for example Breiman[47] p. 45)

$$Q\left[2 \max_{1 \leqslant j \leqslant B} \left(|S_j|\right) > q\right] \leqslant \frac{Q\left[|S_B| > q/4\right]}{1 - \sup_{1 \leqslant j \leqslant B}\left(Q\left[|S_B - S_j| > q/4\right]\right)}. \tag{B.6}$$

Using (B.1), we have

$$Q\left[|S_B| > q/4\right] \leqslant 2\exp\left\{-\log K\left(1 + s - \mathcal{O}\left((\log K)/B\right)^{1/2}\right)\right\}. \tag{B.7}$$

Similarly, using (B.2), for all $K > 1$ such that $\log K < (1+s)32\sigma^2\kappa^2 B$, we have

$$\sup_{0 \leqslant j \leqslant B} Q\left[|S_B - S_j| > q\right] \leqslant 2K^{-(1-1/64-\mathcal{O}((\log K)/B)^{1/2})}. \tag{B.8}$$

Therefore, inserting (B.7) and (B.8) in (B.6) we get for all $K > 1$ such that $\log K < (1+s)32\sigma^2\kappa^2 B$

$$Q\left[2 \max_{1 \leqslant j \leqslant B} \left(|S_j|\right) > ((1+s)32\sigma^2 B \log K)^{1/2}\right] \\ \leqslant 2K^{-(1+s-\mathcal{O}((\log K)/B)^{1/2})}\left(1+\mathcal{O}\left(H_{K,B}\right)\right), \tag{B.9}$$

where $H_{K,B} = K^{-(1-1/64-\mathcal{O}(\log K/B)^{1/2})}$. Inserting (B.9) in (B.5) and noticing that $(1-x)^a \geqslant 1-ax$ for all $0 \leqslant x \leqslant 1$ and $a \geqslant 1$ we get (B.3).

Now we prove (B.4), let $L>1$, $B>1$ an integer and $K>1$ such that $[L]+1 = K \times B$, we have $[K]\times B \leqslant [L]+1 \leqslant ([K]+1)\times B$, we remark that

$$\max_{-[L]-1\leqslant m\leqslant [L]+1} \max_{m\leqslant l\leqslant m+B} \max_{m\leqslant j\leqslant m+B} \left(\left|S_l - S_j\right|\right) \tag{B.10}$$

$$\leqslant \max_{0\leqslant q\leqslant B} \max_{-[K]-1\leqslant i\leqslant [K]-1} \max_{iB+q\leqslant l\leqslant (i+1)B+q} \max_{iB+q\leqslant j\leqslant (i+1)B+q} \left(\left|S_l - S_j\right|\right), \tag{B.11}$$

therefore we have

$$Q\left[\max_{-L\leqslant m\leqslant L} \max_{m\leqslant l\leqslant m+B} \max_{m\leqslant j\leqslant m+B} \left(\left|S_l - S_j\right|\right) > ((1+s)32\sigma^2 B\log K)^{1/2}\right] \tag{B.12}$$

$$\leqslant (B+1)\times Q\left[\max_{-[K]-1\leqslant i\leqslant [K]-1} \max_{iB\leqslant l\leqslant (i+1)B} \max_{iB\leqslant j\leqslant (i+1)B} \left(\left|S_l - S_j\right|\right)\right.$$
$$\left. > ((1+s)32\sigma^2 B\log K)^{1/2}\right]. \tag{B.13}$$

Using (B.3) we obtain (B.4). ∎

Lemma B.3. Assume that for all $\kappa \in]0, k^+[$ (2.4) holds, for all integer $L>0$ and all $D>0$ we have

$$Q\left[\max_{-L\leqslant i\leqslant L} (\beta_i/\alpha_i) \leqslant D^{6/\kappa}\right] \geqslant 1 - D^{-6}(2L+1)\mathbb{E}_Q\left[e^{\kappa\epsilon_0}\right], \tag{B.14}$$

$$Q\left[\max_{-L\leqslant i\leqslant L} (\alpha_i/\beta_i) \leqslant D^{6/\kappa}\right] \geqslant 1 - D^{-6}(2L+1)\mathbb{E}_Q\left[e^{-\kappa\epsilon_0}\right], \tag{B.15}$$

moreover if $D > 2^{1+\kappa/6}$

$$Q\left[\max_{-L\leqslant i\leqslant L} (1/\alpha_i) \leqslant D^{6/\kappa}\right] \geqslant 1 - D^{-6}2^{\kappa}(2L+1)\mathbb{E}_Q\left[e^{\kappa\epsilon_0}\right], \tag{B.16}$$

$$Q\left[\max_{-L\leqslant i\leqslant L} (1/\beta_i) \leqslant D^{6/\kappa}\right] \geqslant 1 - D^{-6}2^{\kappa}(2L+1)\mathbb{E}_Q\left[e^{-\kappa\epsilon_0}\right]. \tag{B.17}$$

Proof. This lemma is a simple consequence of the fact that the random variables $(\alpha_i,\ i\in\mathbb{Z})$ are i.i.d. ∎

Recalling (A.1) and (A.2), we have:

Lemma B.4. Assume (2.2), (2.3), and (2.4). Let $\kappa \in]0, k^+[$, $a>0$, $c>0$ and let us denote $d=a\vee c$. There exists $n_0 \equiv n_0(\sigma, \mathbb{E}[\,|\epsilon_0|^3\,])$ such that for all $n>n_0$, $L>\frac{(2(d\log n))^2}{\sigma^2}+1$ and $D>1$ we have

$$Q\left[U_a^- \wedge U_c^+ > L\right] \leqslant 2q_1^{\frac{L\sigma^2}{(2(d\log n))^2+\sigma^2}}, \tag{B.18}$$

$$Q\left[U_a^- < U_c^+\right] \leqslant \frac{1}{c+a}\left(c+\frac{H_d}{\log n}\right), \tag{B.19}$$

$$Q\left[U_a^- > U_c^+\right] \leqslant \frac{1}{c+a}\left(a+\frac{H_d}{\log n}\right). \tag{B.20}$$

where $q_1 = 0.7 + \frac{3,75\mathbb{E}_Q\left[|\epsilon_0|^3\right]}{(d\log n)\sigma^2} < 1$ and $H_d = (q_1^{\frac{1}{2}\frac{L\sigma^2}{(2(d\log n))^2+\sigma^2}})/(1-q_1) + (6\log D)/\kappa + (L^{3/2}(C)^{1/2}\sigma)/D^3$.

Proof. We have

$$\begin{aligned} Q\left[U_a^- \wedge U_c^+ > L\right] &\leqslant Q\left[U_d^- \wedge U_d^+ > L\right] \\ &= Q\left[\max_{0\leqslant l\leqslant L}|S_l| < (d\log n)\right]. \end{aligned} \tag{B.21}$$

Let $b=\left[\frac{(2(d\log n))^2}{\sigma^2}\right]+1$, for all $L>b$ there exists $k\equiv k(b,L)$ such that $k\times b \leqslant L \leqslant b\times(k+1)$, let us denote $[k]$ the integer part of k, we easily get that

$$Q\left[U_a^- \wedge U_c^+ > L\right] \leqslant \left(Q\left[\left|\frac{S_b}{\sigma b^{1/2}}\right| < \frac{2(d\log n)}{\sigma b^{1/2}}\right]\right)^{[k]}. \tag{B.22}$$

Now we use the Berry–Essen theorem (see Chow and Teicher, p. 299),[48] we get

$$Q\left[\left|\frac{S_b}{\sigma b^{1/2}}\right| < \frac{2(d\log n)}{\sigma b^{1/2}}\right] \leqslant 2\int_0^1 \frac{e^{-x^2}}{\sqrt{2\pi}}dx + \frac{3,75\mathbb{E}_Q\left[|\epsilon_0|^3\right]}{(d\log n)\sigma^2}. \tag{B.23}$$

Moreover, $2\int_0^1 \frac{e^{-x^2}}{\sqrt{2\pi}}dx < 0.7$, therefore, using (B.22) and (B.23) we get (B.18).

To prove (B.19) we use Wald's identity (see Neveu[49]) for the martingale $(S_t^n,\ t\in\mathbb{R})$ and the regular stopping time $U=U_a^-\wedge U_c^+$. Using that $\mathbb{E}_Q\left[S_U^n\right]=0$ and $\mathbb{E}_Q\left[\left(S_{U_a^-}^n+a\right)\mathbb{I}_{U_a^-<U_c^+}\right]\leqslant 0$ we get that

$$Q\left[U_a^-<U_c^+\right]\leqslant \frac{c}{c+a}+\frac{1}{c+a}\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-}\right]. \tag{B.24}$$

We have

$$\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-}\right]=\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U\geqslant[L]+1}\right]$$
$$+\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U<[L]+1}\right]. \tag{B.25}$$

For the second term on the right-hand side of (B.25), noticing that $(S_i^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U=i}\leqslant\frac{\epsilon_i}{\log n}\mathbb{I}_{U_c^+\leqslant U_a^-,U=i}$ we have

$$\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U<[L]+1}\right]\leqslant\frac{1}{\log n}\sum_{i=1}^{[L]}\mathbb{E}_Q\left[(\epsilon_i)\mathbb{I}_{U_c^+\leqslant U_a^-,U=i}\right]. \tag{B.26}$$

For all $D>1$, we have

$$\frac{1}{\log n}\sum_{i=1}^{[L]}\mathbb{E}_Q\left[(\epsilon_i)\mathbb{I}_{U_c^+\leqslant U_a^-,U=i}\right]=\frac{1}{\log n}\sum_{i=1}^{[L]}\mathbb{E}_Q\left[(\epsilon_i)\mathbb{I}_{U_c^+\leqslant U_a^-,U=i,\max_{1\leqslant j\leqslant[L]}(\epsilon_j)\leqslant\frac{6}{\kappa}\log D}\right] \tag{B.27}$$
$$+\frac{1}{\log n}\sum_{i=1}^{[L]}\mathbb{E}_Q\left[(\epsilon_i)\mathbb{I}_{U_c^+\leqslant U_a^-,U=i,\max_{1\leqslant j\leqslant[L]}(\epsilon_j)>\frac{6}{\kappa}\log D}\right] \tag{B.28}$$
$$\leqslant\frac{6\log D}{\kappa\log n}+\frac{\sigma[L]}{\log n}\left(Q\left[\max_{1\leqslant j\leqslant[L]}(\epsilon_j)>\frac{6}{\kappa}\log D\right]\right)^{1/2}, \tag{B.29}$$

where we have used that for the sum in the right hand side of (B.27) the ϵ_i are bounded by $\frac{6}{\kappa}\log D$ and for the sum (B.28) the Cauchy–Schwarz inequality. To end we use (B.14), for all $D>2^{1+\kappa/6}$

$$\mathbb{E}_Q\left[(S_{U_c^+}^n-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U<[L]+1}\right]\leqslant\frac{6\log D}{\kappa\log n}+\frac{\sigma([L])^{3/2}\left(\mathbb{E}_Q\left[e^{\kappa\log\left(\frac{\beta_0}{\alpha_0}\right)}\right]\right)^{1/2}}{D^3\log n}. \tag{B.30}$$

For the first term of the right-hand side of (B.25), using Cauchy–Schwarz inequality we get

$$\mathbb{E}_Q\left[(S^n_{U_c^+}-c)\mathbb{I}_{U_c^+\leqslant U_a^-,U\geqslant [L]+1}\right]\leqslant \frac{\sigma}{\log n}\sum_{i=[L]+1}^{\infty}(Q\left[U\geqslant i\right])^{1/2}, \tag{B.31}$$

then, to estimate, $Q\left[U\geqslant i\right]$ we use (B.18). Collecting what we did above we get (B.19). ∎

Remark B.5. For all $a>0$, $b>0$ and $l>0$ we have

$$Q\left[U_c^+>l\right]\leqslant Q\left[U_c^+\wedge U_a^->l\right]+Q\left[U_c^+>U_a^-\right]. \tag{B.32}$$

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REFERENCES

1. F. Solomon, Random walks in random environment. *Ann. Probab.* **3**(1):1–31 (1975).
2. H. Kesten, M. V. Kozlov, and F. Spitzer, A limit law for random walk in a random environment, *Comp. Math.* **30**:145–168 (1975).
3. Ya. G. Sinai, The limit behaviour of a one-dimensional random walk in a random medium, *Theory Probab. Appl.* **27**(2):256–268 (1982).
4. A. O. Golosov, Localization of random walks in one-dimensional random environments, *Commun. Math. Phys.* **92**:491–506 (1984).
5. A. O. Golosov, Limit distributions for random walks in random environments, *Soviet Math. Dokl.* **28**:18–22 (1986).
6. H. Kesten, The limit distribution of Sinai's random walk in random environment, *Physica* **138A**:299–309 (1986).
7. P. Deheuvels and P. Révész, Simple random walk on the line in random environment, *Probab. Theory Related Fields* **72**:215–230 (1986).
8. P. Révész, *Random Walk in Random and Non-random Environments* (World Scientific, 1989).
9. A. Greven and F. Hollander, Large deviation for a walk in random environment, *Ann. Probab.* **27**(4):1381–1428 (1994).
10. O. Zeitouni and N. Gantert, Quenched sub-exponential tail estimates for one-dimentional random walk in random environment, *Comm. Math. Phys.* **194**:177–190 (1998).
11. A. Pisztora and T. Povel, Large deviation principle for random walk in a quenched random environment in the low speed regime, *Ann. Probab.* **27**:1389–1413 (1999).

12. O. Zeitouni, A. Pisztora, and T. Povel, Precise large deviation estimates for a one-dimensional random walk in a random environment, *Probab. Theory Related Fields* **113**:191–219 (1999).
13. F. Comets, O. Zeitouni, and N. Gantert, Quenched annealed and functional large deviations for one-dimensional random walk in random environment, *Probab. Theory Related Fields* **118**:65–114 (2000).
14. O. Zeitouni, Lectures notes on random walks in random environment, *St Flour Summer School* (2001).
15. Z. Shi, A local time curiosity in random environment, *Stochastic Process. Appl.* **76**2:231–250 (1998).
16. Y. Hu and Z. Shi, The limits of Sinai's simple random walk in random environment, *Ann. Probab.* **26**4:1477–1521 (1998a).
17. Y. Hu and Z. Shi, The local time of simple random walk in random environment, *J. Theor Probab.* **113**:765–793 (1998b)
18. Y. Hu, The logarithmic average of Sinai's Walk in random environment *Period. Math. Hungar.* **41**:175–185 (2000a).
19. Y. Hu, Tightness of localization and return time in random environment, *Stochastic Process. Appl.* **86**(1):81–101 (2000b).
20. Y. Hu and Z. Shi, The problem of the most visited site in random environment, *Probab. Theory Related Fields* **116**(2):273–302 (2000).
21. S. Schumacher, Diffusions with random coefficients, *Contemp. Math.* **41**:351–356 (1985).
22. T. Brox, A one-dimensional diffusion process in a Wiener medium, *Ann. Probab.* **14**(4):1206–1218 (1986).
23. Z. Shi, Sinai's walk via stochastic calculus, *Panoramas Synthéses* **12**:53–74 (2001).
24. A. Dembo, A. Guionnet, and O. Zeitouni, Aging properties of Sinai's model of random walk in random environment, in *St. Flour Summer School 2001, Springer's Lecture Notes in Mathematics*, Vol. 1837.
25. F. Comets and S. Popov, Limit law for transition probabilities and moderate deviations for Sinai's random walk in random environment, *Preprint*, (2003).
26. N. Gantert and Z. Shi, Many visits to a single site by a transient random walk in random environment, *Stochastic Process. Appl.* **99**:159–176 (2002).
27. H. Tanaka, Localization of a diffusion process in a one-dimensional brownian environmement, *Commun Pure Appl. Math.* **17**:755–766 (1994).
28. P. Mathieu, Limit theorems for diffusions with a random potential, *Stochastic Process. Appl.* **60**:103–111 (1995).
29. H. Tanaka, Limit theorem for a brownian motion with drift in a white noise environment, *Chaos Solitons Fractals* **11**:1807–1816 (1997).
30. H. Tanaka and K. Kawazu, A diffusion process in a brownian environment with drift, *J. Math. Soc. Japan* **49**:189–211 (1997).
31. P. Mathieu, On random perturbations of dynamical systems and diffusion with a random potentiel in dimension one, *Stochastic Process. Appl.* **77**:53–67 (1998).
32. M. Taleb, Large deviations for a brownian motion in a drifted brownian potential, *Ann. Probab.* **29**(3):1173–1204 (2001).
33. S. A. Kalikow, Generalised random walk in random environment, *Ann. Prob.* **9**(5):753–768 (1981).
34. V. V. Anshelevich, K. M. Khanin, Ya. G. Sinai, Symmetric random walks in random environments, *Commun. Math. Phys.* **85**:449–470 (1982).
35. R. Durrett, Some multidimensional rwre with subclassical limiting behavior, *Commun. Math. Phys.* **104**:87–102 (1986).

36. J. P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, Anomalous diffusion in random media of any dimensionality, *J. Phys.* **48**:1445–1450 (1987).
37. J. Bricmont and A. Kupiainen, Random walks in asymetric random environments, *Commun Math. Phys.* **142**:342–420 (1991).
38. A. S. Sznitman, Lectures on random motions in random media, *Preprint* (1999).
39. A. S. Sznitman, On new examples of ballistic random walks in random environment, *Ann. Probab.***31**(1):285–322 (2003).
40. S. R. S. Varadhan, Large deviations for random walks in random environment, *Commun. Pure Appl. Math.* **56**(8): 1222–1245 (2003).
41. F. Rassoul-Agha, The point of view of the particule on the law of large numbers for random walks in a mixing random environment, *Ann. Probab.* **31**:1441–1463 (2003).
42. F. Comets and O. Zeitouni, A law of large numbers for random walk in random environments, *To appear in Ann. Probab.* (2004).
43. K. L. Chung, *Markov Chains* (Springer-Verlag, 1967).
44. M. Cassandro, E. Orlandi, P. Picco, and M. E. Varés, One dimensional random field Kac's model: localisation of the phases, *Preprint* (2004+).
45. L. LeCam, *Asymptotic Methods in Statistical Decision Theory* (Springer-Verlag, 1986).
46. A. Renyi, *Probability Theory* (North-Holland Publishing Company, 1970).
47. L. Breiman, *Probability* (Addison-Wesley Publishing Company, Inc, 1968).
48. Y. S. Chow and H. Teicher, *Probability Theory* 3rd edn. (Srpinger, 1997).
49. J. Neveu, *Martinguales à temps Discret* (Masson et Cie 1972).